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SOME PROPERTIES OF THE FUNDAMENTAL POLYNOMIALS OF BERNSTEIN-SCHURER

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Abstract. We prove that the fundamental polynomials of Bernstein-Schurer, defined at (1.4) have properties similar to the properties of the fundamental polynomials of Bernstein.

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1. Preliminaries.

It is well known (see for example Agratini O., [1]) that the fundamental polynomials of Bernstein

$$p_{m,k}(x) = \sum_{k=0}^{m} \binom{m}{k} x^k (1-x)^{m-k} \quad (1.1)$$

have the properties expressed in the following:

Theorem 1.1.([1], pp. 80-81). *Let m and s be positive integers and let $T_{m,s}(x)$ be the polynomials*

$$\tilde{T}_{m,s}(x) = \sum_{k=0}^{m} (k - mx)^s \, p_{m,k}(x) \quad (1.2)$$

Then:

1^0. $T_{m,s+1}(x) = x(1-x)\left\{T'_{m,s}(x) + msT_{m,s-1}(x)\right\}, \forall x \in [0,1]$,where $T'_{m,s}(x)$ denotes the first order derivative of $T_{m,s}(x)$;

2^0. $T_{m,2}(x) = mx(1-x)$;

$T_{m,3}(x) = m(1-2x) \times (1-x)$;

$T_{m,4}(x) = 3m^2x^2(1-x) + m\left\{x(1-x) - 6x^2(1-x)^2\right\}$.

3^0. For any gived $\delta > 0$,

$$\lim_{m\to\infty} m \sum_{\left|\frac{k}{m}-x\right|\geq\delta} \left(\frac{k}{m}-x\right)^2 p_{m,k}(x) = 0.$$

Let p be a given non-negative integer. F. Schurer (see[2]) introduced and studied the operators $\tilde{B}_{m,p} : C([0,1+p]) \to C([0,1])$ defined for any function $f \in C([0,1+p])$ by

$$\left(\tilde{B}_{m,p} f\right)(x) = \sum_{k=0}^{m+p} \tilde{p}_{m,k}(x)\, f\left(\frac{k}{m}\right). \tag{1.3}$$

In (1.3) $\tilde{p}_{m,k}(x)$ denotes the fundamental polynomials of Bernstein- Schurer, defined as follows:

$$\tilde{p}_{m,k}(x) = \sum_{k=0}^{m+p} \binom{m+p}{k} x^k (1-x)^{m+p-k}. \tag{1.4}$$

Clearly, for $p = 0$ the operator (1.3) reduces to the classical operator of Bernstein and the polynomials (1.4) are in this case the fundamental polynomials of Bernstein (1.1).

Our purpose is to established an analogous of the Theorem 1.1 for the polynomials

$$\tilde{T}_{m,s}(x) = \sum_{k=0}^{m+p} \{k - (m+p)x\}^s\, \tilde{p}_{m,k}(x). \tag{1.5}$$

We will use the properties of the Bernstein-Schurer operator, contained in the following

Theorem 1.2. ([1],[2]) Let $e_i(x) = x^i$ $(1 = 0,1,2)$ *be test functions. The following equalities*

(i) $\left(\tilde{B}_{m,p} e_0\right)(x) = 1;$

(ii) $\left(\tilde{B}_{m,p} e_1\right)(x) = (1 + p/m)x;$

(iii) $\left(\tilde{B}_{m,p} e_2\right)(x) = (m+p) \cdot m^{-2} \left\{(m+p)x^2 + x(1-x)\right\}$

hold, for any $x \in [0, 1+p]$.

2. Main results

Theorem 2.1. *Let* m *and* s *be positive integers and let* $\tilde{T}_{m,s}$ *be the polynomials defined at (1.5).*

Then

$$\tilde{T}_{m,s+1}(x) = x(1-x)\left\{T'_{m,s}(x) + (m+p)s\,\tilde{T}_{m,s-1}(x)\right\}$$

for any $x \in [0, 1+p[$.

Proof. For $x = 0$, we have $\tilde{T}_{m,s}(0) = 0$ and (2.1) holds. For $x \in]0, +p+1[$, taking into account (1.5), we get:

$$\widetilde{T}'_{m,s}(x) = -(m+p)s\,\widetilde{T}_{m,s-1}(x) + \sum_{k=0}^{m+p} \{k-(m+p)x\}^s \,\widetilde{p}'_{m,k}(x) \tag{2.1}$$

where $\widetilde{p}'_{m,s}(x)$ denotes the first order derivative of $\widetilde{p}_{m,k}(x)$. But these derivative are expressed by

$$\widetilde{p}'_{m,k}(x) = \binom{m+p}{k} x^{k-1}(1-x)^{m+p-k-1}\{k-(m+p)x\}.$$

Replacing this expression in (2.2), we get

$$\widetilde{T}'_{m,s}(x) = -(m+p)s\,\widetilde{T}_{m,s-1}(x) + \{x(1-x)^{-1}\}\,\widetilde{T}_{m,s+1}(x)$$

which is equivalent with (2.1) and the proof ends.

Theorem 2.2. *The following identities*

$$\widetilde{T}_{m,2}(x) = (m+p)x(1-x); \tag{2.2}$$

$$\widetilde{T}_{m,3}(x) = (m+p)x(1-x)(1+2p-2x); \tag{2.3}$$

$$\widetilde{T}_{m,4}(x) = 3(m+p)^2x^2(1-x)^2 + (m+p)x(1-x)\times \tag{2.4}$$

$$\times\{(1+2p)(1-2x) - 2x(1-x) + 6(m+p)x(1-x)(p-x)\}$$

hold, for any $x \in [0, p+1[$.

Proof. Applying the Theorem 1.2, we get

$$\widetilde{T}_{m,0}(x) = 1, \quad \widetilde{T}_{m,1}(x) = p, \quad (\forall)x \in [0, p+1].$$

Next, using the Theorem 2.1, we arrive to the desired identities.

Theorem 2.3. *For any gived* $\delta > 0$, *the equality*

$$\lim_{m\to\infty}(m+p)\sum_{\left|\frac{k}{m+p}-x\right|\geq\delta}\left(\frac{k}{m+p}-x\right)^2\widetilde{p}_{m,k}(x) = 0 \tag{2.5}$$

holds.

Proof. Because $\left|\dfrac{k}{m+p}-x\right| \geq \delta$ it follows that $\left(\dfrac{k}{m+p}-x\right)^2 \geq \delta^2$ and $\left(\dfrac{k}{m+p}-x\right)^2\delta^{-2} \geq 1$. Next, we can write:

$$\sum_{\left|\frac{k}{m+p}-x\right|\geq\delta}\left(\frac{k}{m+p}-x\right)^2\widetilde{p}_{m,k}(x) \leq \delta^{-2}\sum_{\left|\frac{k}{m+p}-x\right|\geq\delta}\left(\frac{k}{m+p}-x\right)^4\widetilde{p}_{m,k}(x) \leq$$

$$\leq \delta^{-2} \sum_{k=0}^{m+p} \left(\frac{k}{m+p} - x \right)^4 \widetilde{p}_{m,k}(x) = (m+p)^{-4} \cdot \delta^{-2} \cdot \widetilde{T}_{m,4}(x).$$

It follows the inequality:

$$(m+p) \sum_{\left| \frac{k}{m+p} - x \right| \geq \delta} \left(\frac{k}{m+p} - x \right)^2 \widetilde{p}_{m,k}(x) \leq \frac{1}{(m+p)^3 \delta^2} \cdot \widetilde{T}_{m,4}(x). \tag{2.6}$$

Taking into account (2.4) we get

$$\lim_{m \to \infty} \frac{1}{(m+p)^3 \delta^2} \widetilde{T}_{m,4}(x) = 0,$$

and then, using (2.6) we arrive to (2.5).

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