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THE VORONOVSKAJA THEOREM FOR BERNSTEIN-SCHURER OPERATORS

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Abstract The Bernstein - Schurer operators are considered and for these kind of operators a Voronskaja type theorem is established. The main result of the paper is the Theorem 2.1.

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1. Preliminaries.

Let $B_m : C([0,1]) \to C([0,1])$ be the Bernstein operators, defined for any $f \in C([0,1])$ by

$$(B_m f)(x) = \sum\nolimits_{k=0}^{m} p_{m,k}(x) \cdot f\left(\frac{k}{m}\right) \tag{1.1}$$

where $p_{m,k}(x)$ are the fundamental polynomials of Bernstein, defined as follows:

$$p_{m,k}(x) = \binom{m}{k} x^k (1-x)^{m-k} \tag{1.2}$$

In 1932 Voronovskaia., E., (see[4]), proved the result contained in the following theorem.

Theorem 1.1 *([4]). Let $f \in C([0,1])$ be a function which has the second order derivative in the point $x \in [0,1]$. Then the equality*

$$\lim_{m\to\infty} m((B_m f)(x) - f(x)) = \frac{x(1-x)}{2} f''(x), \tag{1.3}$$

holds.

Considering the non-negative integer p, F. Schurer (see [3]) introduced and studied in 1962 the operators $\widetilde{B}_{m,p} : C([0,1+p]) \to C([0,1])$, defined for any function $f \in C([0,1+p])$ by: $\left(\widetilde{B}_{m,p} f\right)(x) = \sum_{k=0}^{m+p} \widetilde{p}_{m,k}(x)\, f\left(\frac{k}{m}\right)$.In (1.4), $\widetilde{p}_{m,k}(x)$ denotes the fundamental polynomials of Bernstein - Schurer, defined as follows:

$$\widetilde{p}_{m,k}(x) = \binom{m+p}{k} x^k (1-x)^{m+p-k} \tag{1.5}$$

In our earlier paper ([2]), we studied some properties of the fundamental polynomials of Bernstein-Schurer. These properties are contained in the following theorems.

Theorem 1.2.([2], Th. 2.1). *Let m and s be positive integers and $\widetilde{T}_{m,s}(x)$ be the polynomials*

$$\widetilde{T}_{m,s}(x) = \sum_{k=0}^{m+p} (k-(m+p)x)^s \, \widetilde{p}_{m,k}(x).$$

The polynomials $\widetilde{T}_{m,s}(x)$ satisfy the relation

$$\widetilde{T}_{m,s+1}(x) = x(1-x)\left\{T'_{m,s}(x) + (m+p)s\, \widetilde{T}_{m,s-1}(x)\right\}$$

for any $x \in [0, 1+p]$, where $T'_{m,s}(x)$ denotes the first order derivative of $T_{m,s}$.

Theorem 1.3. ([2], Th. 2.2) *For the polynomials $\widetilde{T}_{m,s}(x)$ the equalities*

(i) $\widetilde{T}_{m,2}(x) = (m+p)\, x(1-x)$; (ii) $\widetilde{T}_{m,3}(x) = (m+p)\, x(1-x)(1+2p-2x)$;

(iii) $\widetilde{T}_{m,4}(x) = 3(m+p)^2 x^2 (1-x)^2 + (m+p)\, x(1-x) \times$
$\times \{(1+2p)(1-2x) - 2x(1-x) + 6(m+p)x(1-x)(p-x)\}$ *hold for any* $x \in [0, 1+p]$.

Theorem 1.4. ([2], Th. 2.3) *For any gived $\delta > 0$ the equality*

$$\lim_{m\to\infty} (m+p) \cdot \sum_{\left|\frac{k}{m+p}-x\right| \geq \delta} \left(\frac{k}{m+p} - x\right)^2 \cdot \widetilde{p}_{m,k}(x) = 0$$

holds. Use the above results we shall prove the Voronovkaja's theorem for the Bernstein-Schurer operators (1.4).

2. The main result.

First, we establish the auxilliary result contained in the following lemma.

Lemma 2.1. *Let $\psi_x : [0, 1+p] \to \mathbb{R}$, $\psi_x(t) = t - x$. Then, the following equalities*

$$\left(\widetilde{B}_{m,p}\psi_x\right)(x) = (p/m)\, x\,; \tag{2.1}$$

$$\left(\widetilde{B}_{m,p}\psi_x^2\right)(x) = (p^2/m^2)\, x^2 + ((m+p)/m^2) \times (1-x), \tag{2.2}$$

hold, for any $x \in [0, 1+p]$.

Proof. It is well known (see Schurer., F., [3]) that the operator $\widetilde{B}_{m,p}$ has the properties $\left(\widetilde{B}_{m,p}e_0\right)(x) = 1$, $\left(\widetilde{B}_{m,p}l_1\right)(x) = (1+p/m)x$, $\left(\widetilde{B}_{m,p}e_2\right)(x) = (m+p)\,m^{-2}\{(m+p)x^2 + x(1-x)\}$, where $e_i(x) = x^i$ $(i = 0, 1, 2)$ are the test functions. Using these properties and the linearity of the Bernstein - Schurer operators, we get (2.1) and (2.2). Now, we are ready to prove the main result of the paper, which is the following Voronovskaja type theorem for the Bernstein-Schurer operators.

Theorem 2.1. Let $f \in C([0, 1+p])$ *be a function two times derivable at the point* $x \in [0, 1+p]$. *Then the equality*

$$\lim_{m\to\infty}(m+p)\left\{\left(\widetilde{B}_{m,p}f\right)(x)-f(x)\right\}=p\cdot f'(x)+\frac{x(1-x)}{2}f''(x) \tag{2.3}$$

holds.

Proof. The Taylor's expansion of f in a neighborhood of $x\in[0,1+p]$ leads us to: $f(t)=f(x)+(t-x)f'(x)+\frac{(t-x)^2}{2}f''(x)+(t-x)^2\mu(t-x)$.Note that in (2.2) μ denotes a bounded function having the property $\lim_{h\to 0}\mu(h)=0$. After some transformations of (2.2) we arrive to:

$$\left(\widetilde{B}_{m,p}f\right)(x)=f(x)+f'(x)\cdot\left(\widetilde{B}_{m,p}\psi_x\right)(x)+$$

$$+\frac{f''(x)}{2}\left(\widetilde{B}_{m,p}\psi_x^2\right)(x)+\sum_{k=0}^{m+p}(k/m-x)^2\mu(k/m-x)\,\widetilde{p}_{m,k}(x). \tag{2.4}$$

Using the Lemma 2.1, from (2.3) we get

$$(m+p)\left\{\widetilde{B}_{m,p}f(x)-f(x)\right\}=$$

$$=\frac{p(m+p)}{m}f'(x)+\frac{f''(x)}{2}\left\{\frac{p^2(m+p)}{m^2}x^2+\frac{(m+p)^2}{m^2}x(1-x)\right\}+ \tag{2.5}$$

$$+(m+p)\sum_{k=0}^{m+p}(k/m-x)^2\mu(k/m-x)\,\widetilde{p}_{m,k}(x).$$

If $R_m(x)$ denotes the sum $R_m(x)=\sum_{k=0}^{m+p}(m+p)\cdot(k/m-x)^2\mu(k/m-x)\,\widetilde{p}_{m,k}(x)$, we shall prove that $\lim_{m\to\infty}R(x)=0$. Let $\varepsilon>0$ be arbitrarly given. Because $\lim_{h\to 0}\mu(h)=0$ it follows that there exists a $\delta>0$ so that for any h having the property $|h|<\delta$ the inequality

$$|\mu(h)|<\varepsilon \tag{2.6}$$

holds.

The set $I=\{0,1,\ldots,m+p\}$ can be decomposed in the form

$$I=\left\{k\in I:\left|\frac{k}{m}-x\right|<\delta\right\}\cup\left\{k\in I:\left|\frac{k}{m}-x\right|\geq\delta\right\}. \tag{2.7}$$

Using the above representation of I, we get

$$|R_m(x)| = \left|\sum_{k=0}^{m+p}(m+p)(k/m-x)^2\cdot\mu(k/m-x)\cdot\widetilde{p}_{m,k}(x)\right|\leq$$

$$\leq (m+p)\varepsilon\sum_{k\in I_1}(k/m-x)^2\widetilde{p}_{m,k}(x)+$$

$$+(\sup\mu)\sum_{k\in I_2}(m+p)\cdot(k/m-x)^2\widetilde{p}_{m,k}(x) \tag{2.8}$$

For the first sum of the right side of (2.8) we can write

$$\sum_{k\in I_1}(k/m-x)^2\widetilde{p}_{m,k}(x)\le\sum_{k\in I_1}1\cdot\widetilde{p}_{m,k}(x)\le\sum\nolimits_{k=0}^{m+p}\widetilde{p}_{m,k}(x)=1 \quad (2.9)$$

Applying the Theorem 1.4, we get that the limit of the second sum of the right side of (2.8) is zero, i.e. there exists $m_0\in\mathbb{N}$ so that for any $m\ge m_0$ the inequality

$$\sum_{k\in I_2}(m+p)\cdot(k/m-x)^2\widetilde{p}_{m,k}(x)<\varepsilon\cdot(\sup\mu)^{-1} \quad (2.10)$$

Using now (2.8), (2.9), and (2.10), we can conclude that there exists $m_0\in\mathbb{N}$ so that for any $m\ge\mathbb{N}$, $m\ge m_0$ the inequality

$$|R_m(x)|\le(m+p)\varepsilon+\varepsilon=(m+p+1)\varepsilon \quad (2.11)$$

holds.

In the (2.11) we choose $\varepsilon=\{m(m+p+1)\}^{-1}$. Then, it follows that there exists $m_0\in\mathbb{N}$ so that for any $m\ge\mathbb{N}$, $m\ge m_0$ the inequality

$$|R_m(x)|\le\{m(m+p)\}^{-1}\cdot(m+p+1)=m^{-1} \quad (2.12)$$

holds. From (2.12) it follows that $\lim\limits_{m\to\infty}R_m(x)=0$ and the proof ends.

REFERENCES

[1] Agratini, O., *Aproximare prin operatori liniari*, Presa Universitară Clujeană, 2000(in Romanian)

[2] Bărbosu, D., *Some properties of the fundamental polynomials of Bernstein-Schurer* (submitted)

[3] Schurer, F., *Linear positive operators in approximation theory,* Math. Inst. Techn. Univ. Delft, Report, 1962

[4] Voronovskaja, E., *Determination de la forme asymptotique d'approximation des functions par les polynomes de M. Bernstein,* C.R. Acad. Sci. URSS(1932), 79-85

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