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ON SEMIGROUPS CORRESPONDING TO NON-LOCAL BOUNDARY CONDITIONS WITH APPLICATIONS TO SYSTEM THEORY

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Abstract. A class of infinitesimal generators, motivated by some non-local problem, is introduced. An abstract version of a system of Riccati type equations for quadratic cost control problem and for linear systems related to the introduced semigroups are obtained as well.

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1. Semigroups and abstract boundary conditions. General results

Let X and Y be Banach spaces. Let A be the infinitesimal generator of a strongly continuous semigroup $(T_t)_{t\geq 0}$ on Y and let B, C, F be bounded operators:

$$B : X \to X, \quad C : Y \to X, \quad F : X \to Y.$$

One of the main results of this section is the following theorem.

Theorem 1. *Let $\mathcal{A}$ be an operator defined on*

$$\mathcal{D}(\mathcal{A}) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : y - Fx \in \mathcal{D}(A) \right\} \tag{1}$$

by the formula

$$\mathcal{A}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} Bx + Cy \\ A(y - Fx) \end{pmatrix}. \tag{2}$$

Then

(i) *$\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup $(\mathcal{T}_t)_{t\geq 0}$ on $\mathcal{E} = X \times Y$.*

(ii) *If $(T_t)_{t\geq 0}$ is a holomorphic semigroup then the semigroup $(\mathcal{T}_t)_{t\geq 0}$ is holomorphic as well .*

Theorem 2. *Let $\mathcal{E}$ be a reflexive Banach space and $\mathcal{A}$ the infinitesimal generator of a strongly continuous semigroup on $\mathcal{E}$. Define $\mathcal{F} = \mathcal{I} - \mathcal{P}$. Then the operator $\mathcal{AP} + \mathcal{R}$ with the domain $\mathcal{P}^{-1}\mathcal{D}(\mathcal{A})$, and the operator $\mathcal{PA} + \mathcal{R}$ with the domain $\mathcal{D}(\mathcal{A})$ generate semigroups on $\mathcal{E}$ if*

a) *- $\mathcal{A}$ generates a holomorphic semigroup and $\mathcal{F}$ is a compact operator or,*

- $\mathcal{P}$ *is invertible operator and* $\mathcal{F}(\mathcal{E}) \subset \mathcal{D}(\mathcal{A})$, *or if*

b) $\mathcal{E} = \begin{matrix} X \\ \times \\ Y \end{matrix}$, $\mathcal{A} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, $\mathcal{P} = \begin{pmatrix} I & -F \\ 0 & I \end{pmatrix}$, *where X, Y are Banach spaces, A and B generators on X and Y respectively, and F a bounded operator from X into Y. It is assumed that* $\mathcal{D}(\mathcal{A}) = \begin{matrix} \mathcal{D}(A) \\ \times \\ \mathcal{D}(B) \end{matrix}$

Proposition 1. *Let $\mathcal{A}$ generate a holomorphic semigroup on a reflexive space $\mathcal{E}$ and let an operator $\mathcal{F}$ defined on $\mathcal{D}(\mathcal{A})$ be $\mathcal{A}$-compact. Then the operator $\mathcal{A}+\mathcal{F}$ also generates a holomorphic semigroup on $\mathcal{E}$.*

In Propositions 2 and 3 bellow $W^{2,2} = W^{2,2}[0,1;\mathbf{R}^n]$ and the operator $A_0 = d^2/dx^2$ is defined by

$$\mathcal{D}(A_0) = \{y \in W^{2,2};\ y(0) = y(1) = 0\}. \tag{3}$$

It is well known that A_0 is a self-adjoint operator on $L^2[0,1;\mathbf{R}^n]$ and generate a holomorphic semigroup.

If $f(\cdot) \in L^2[0,1;\mathcal{L}(\mathbf{R}^n)]$ then $\int f$ denotes the transformation $\int fy = \int_0^1 f(x)y(x)dx$ from $L^2[0,1;\mathbf{R}^n]$ into $\mathbf{R}^n$.

Proposition 2. *For every function $f \in L^2[0,1;\mathcal{L}(\mathbf{R}^n)]$ the operator $\mathcal{A} = d^2/dx^2$ defined on*

$$\mathcal{D}(\mathcal{A}) = \left\{ y \in W^{2,2};\ y(0) = \int fy,\ y(1) = 0 \right\}, \tag{4}$$

can be decomposed into the form $\mathcal{A} = A_0\mathcal{P}$ where $\mathcal{D}(A_0)$ is defined by (4)and $\mathcal{I} - \mathcal{P}$ is a finite dimensional operator. Thus $\mathcal{A}$ generates a holomorphic semigroup.

Proposition 3. *The operator $\mathcal{A}$ defined by*

$$\mathcal{A}\begin{pmatrix} y \\ y(0) \\ y(1) \end{pmatrix} = \begin{pmatrix} \dfrac{d^2y}{dx^2} \\ B_{11}y(0) + B_{12}y(1) + \int C_1 y \\ B_{21}y(0) + B_{22}y(1) + \int C_2 y \end{pmatrix}, \tag{5}$$

$$\mathcal{D}(\mathcal{A}) = \left\{ \begin{pmatrix} y \\ y(0) \\ y(1) \end{pmatrix},\ y \in W^{2,2} \right\}$$

can be decomposed into the form

$$\mathcal{A} = \begin{pmatrix} A_0 & 0 \\ 0 & \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \end{pmatrix} \mathcal{P} + \mathcal{R},$$

where

$$\mathcal{P} = \begin{pmatrix} I & -F \\ 0 & I \end{pmatrix}, \ \mathcal{R} = \begin{pmatrix} 0 & 0 \\ \begin{pmatrix} \int C_1 \\ \int C_2 \end{pmatrix} & 0 \end{pmatrix},$$

and F is a bounded operator. Thus $\mathcal{A}$ generates a holomorphic semigroup on $\mathcal{E}$.

Next, according to [5, p.183] we give some results concerning the following semilinear initial value problem

$$\begin{cases} \dfrac{du(t)}{dt} + Au(t) = f(t, u(t)), & t > t_0 \\ u(t_0) = u_0 \end{cases} \tag{5}$$

where $-A$ is the infinitesimal generator of a C_0 semigroup $(T_t)_{t \geq 0}$, on a Banach space X and $f : [t_0, T] \times X \to X$ is continuous in t and satisfies a Lipschitz condition in u.

Theorem 3. *Let $f : [t_0, T] \times X \to X$ be continuous in t on $[t_0, T]$ and uniformly Lipschitz continuous (with constant L) on X. If $-A$ is the infinitesimal generator of a C_0 semigroup $(T_t)_{t \geq 0}$, on X then for every $u_0 \in X$ the initial value problem (5) has a unique mild solution $u \in C([t_0, T]; X)$. Moreover, the mapping $u_0 \to u$ is Lipschitz continuous from X into $C([t_0, T]; X)$.*

Theorem 4. *Let $-A$ be the infinitesimal generator of a C_0 semigroup $(T_t)_{t \geq 0}$ on X. If $f : [t_0, T] \times X \to X$ is continuously differentiable from $[t_0, T] \times X$ into X then the mild solution of (5) with $u_0 \in \mathcal{D}(A)$ is a classical solution of the initial value problem.*

2. Applications to infinite dimensional system theory

In this section we illustrate how the results mentioned in the previous section can be used to solve, or to simplify, some problems in system theory.

2.1 Parabolic systems on finite intervals. Dirichlet boundary conditions

In [3] was considered and solved the optimal control problem for linear systems described by the heat equation with control on the boundaries:

$$\frac{\partial y}{\partial t}(t, x) = \frac{\partial^2 y}{\partial x^2}(t, x), \ t > 0, \ x \in (0, 1) \tag{6}$$

$$
\begin{cases}
\dfrac{\partial y}{\partial t}(t,0) = L_0 \dfrac{\partial y}{\partial t}(t,0) + B_0 u(t) \\
\dfrac{\partial y}{\partial t}(t,1) = L_1 \dfrac{\partial y}{\partial t}(t,1) + B_1 u(t), \; t > 0
\end{cases} \tag{7}
$$

with initial data $y(0,0) \in \mathbf{R}^n$, $y(0,1) \in \mathbf{R}^n$, $y(0,\cdot) \in L^2([0,1];\mathbf{R}^n)$. A system of Riccati type equations was obatined and studied. For the Riccati system an existence and uniqueness theorem was proved.

In this paper we present a general model which contains systems like (6), (7) as special cases. Moreover, more general boundary conditions than (7) may be also included. For instance we shall investigate system (8) with non-local boundary conditions like

$$
\begin{cases}
\dfrac{\partial y(t,0)}{\partial t} = L_0 y(t,0) + \displaystyle\int_0^1 C_0(x) y(t,x)dx \\
\dfrac{\partial y(t,1)}{\partial t} = L_1 y(t,1) + \displaystyle\int_0^1 C_1(x) y(t,x)dx.
\end{cases} \tag{8}
$$

We start from a well known result. The operator $A = \dfrac{d^2}{dx^2}$ defined on $\mathcal{D}(A) = \{y \in W^{2,2}[a,b],\, y(a) = y(b) = 0\}$ is the self-adjoint generator of a C_0-semigroup $(T_t)_{t \geq 0}$ on the space $Y = \quad = L^2([a,b],\mathbf{R}^n)$. Let $X = \mathbf{R}^n \times \mathbf{R}^n$ and define B and C as follows:

$$
B(x^1,x^2) = \begin{pmatrix} B_{11}x^1 + B_{12}x^2 \\ B_{21}x^1 + B_{22}x^2 \end{pmatrix}, \; Cy = \begin{pmatrix} \displaystyle\int_a^b C_1(s)y(s)ds \\ \displaystyle\int_a^b C_2(s)y(s)ds \end{pmatrix}
$$

where $B_{i,j}$ are $n \times n$ matrices, $(x^1,x^2) \in \mathbf{R}^n \times \mathbf{R}^n$ and C_1, C_2 are functions from $L^2([a,b],\mathbf{R}^n \times \mathbf{R}^n)$. Moreover, let

$$
F(x^1,x^2) = x^1 + \frac{s-a}{b-s}(x^2 - x^1), \quad s \in [a,b].
$$

Then Theorem 1 easily implies the following proposition.

Proposition 4. *The operator $\mathcal{A}$ defined on the set $\mathcal{D}(\mathcal{A}) = \{(y(a),y(b),y) : y \in W^{2,2}[a,b]\}$ by the formulae*

$$
\mathcal{A}\begin{pmatrix} y(a) \\ y(b) \\ y \end{pmatrix} = \begin{pmatrix} B_{11}y(a) + B_{12}y(b) + \displaystyle\int_a^b C_1(s)y(s)ds \\ B_{21}y(a) + B_{22}y(b) + \displaystyle\int_a^b C_2(s)y(s)ds \\ \dfrac{d^2y}{dx^2} \end{pmatrix}
$$

is the infinitesimal generator of a C_0-semigroup $(T_t)_{t\geq 0}$ on $X \times Y$. Its conjugate $\mathcal{A}^*$ is defined on $\mathcal{D}(\mathcal{A}^*) = \{(x^1, x^2, y); x^1, x^2 \in \mathbf{R}^n, y \in W^{2,2}[a, b:], y(a) = y(b) = 0\}$.

2.2 Parabolic systems on finite intervals. Mixed boundary conditions

The following result is well known. The operator $A = \dfrac{d^2}{dx^2}$ defined on $\mathcal{D}(A) = \{y \in W^{2,2}[a, b:], y'(a) = y(b) = 0\}$ is the self-adjoint generator of a C_0-semigroup $(T_t)_{t\geq 0}$. Let the operators B and C be defined as in 2.1 and define the operator F as follows:

$$F(x^1, x^2) = x^1 x + (x^2 - x^1 b), \quad x \in [a, b:].$$

The following proposition is a straightforward consequence of Theorem 1.

Proposition 5. *The operator $\mathcal{A}$ defined on the set $\mathcal{D}(\mathcal{A}) = \{(y'(a), y(b), y), y \in W^{2,2}[a, b]\}$ by the formulae*

$$\mathcal{A}\begin{pmatrix} \dfrac{dy}{dx}(a) \\ y(b) \\ y \end{pmatrix} = \begin{pmatrix} B_{11}\dfrac{dy}{dx}(a) + B_{12}y(b) + \displaystyle\int_a^b C_1(x)y(x)dx \\ B_{21}\dfrac{dy}{dx}(a) + B_{22}y(b) + \displaystyle\int_a^b C_2(x)y(x)dx \\ \dfrac{d^2y}{dx^2} \end{pmatrix}$$

is the infinitesimal generator of a C_0-semigroup $(T_t)_{t\geq 0}$ on $X \times Y$.

Remark 2. Semigroups constracted in Proposition 2 and Proposition 3 are holomorphic as follows from Theorem 1, (ii).

3. Quadratic cost control problems. Systems of Riccati equations

Let us consider the following controlled system:

$$z(t) = \mathcal{T}_t z(0) + \int_0^t \mathcal{T}_{t-s}\mathcal{B}u(s)ds \tag{9}$$

where $\mathcal{B}$ is a linear bounded operator from a Hilbert space U into $\mathcal{E} = X \times Y$. The "regulator" problem is to find a control $u(\cdot) \in L^2([0, T], U)$ which minimizes the performance index

$$J(u) = \int_0^t (\langle Qz(t), z(t)\rangle_{\mathcal{E}} + \langle Hu(t), u(t)\rangle_U)dt + \langle G_0 z(T), Z(T)\rangle_{\mathcal{E}}, \tag{10}$$

where Q, H, G_0 are self-adjoint operators on $\mathcal{E}$ and U with $Q \geq 0$, $H > 0$, $G_0 \geq 0$.

The optimal control is given, via the Riccati equation:

$$\frac{d}{dt}\langle P(t)x, x_1\rangle = \langle \mathcal{A}x, P(t)x_1\rangle + \langle P(t)x, \mathcal{A}x_1 > + \langle Qx, x_1\rangle - \langle P(t)\mathcal{B}H^{-1}\mathcal{B}^*P(t)x, x_1\rangle,$$

$P(0) = P_0$ for $x, x_1 \in \mathcal{D}(\mathcal{A})$ *in a feedback form* $u(t) = -H^{-1}\mathcal{B}^*P(T-t)z(t)$, $t \geq 0$ *and* $\inf J(u) = \langle P(T)z(0), z(0)\rangle$, *see* [1].

Moreover the Riccati equation has the unique non-negative solution $P(t)$, $t \geq 0$ see [1].

The problem considered above is in fact the problem of the linear regulator for the infinite dimensional system

$$\dot{z} = \mathcal{A}z + \mathcal{B}u. \tag{11}$$

The solution of this problem can be expressed through the solution $P = P(t)$, $t \geq 0$ of the inner product Riccati equation, see [6]. If the operator $\mathcal{A}$ can be decomposed into the form $\mathcal{A} = \mathcal{A}_0\mathcal{P}$ or $\mathcal{A} = \mathcal{L}\mathcal{A}_0\mathcal{P}$ then the Riccati equation can be written in the terms of the operator $\mathcal{A}_0$ and its domain $\mathcal{D}(\mathcal{A}_0)$ only. Let us take, for instance, into account the situation where the "regulation" is implemented through a new dynamical system. This is the case of the controlled systems considered by the author in [3]. The state space is then of the form $\mathcal{E} = \begin{matrix} X \\ \times \\ Y \end{matrix}$ and the generator $\mathcal{A}$ is given, according to Proposition 3 and Proposition 4, by

$$\mathcal{A} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} I & -F \\ 0 & I \end{pmatrix} + \mathcal{R}.$$

After standard transformations we obtain the inner product Riccati equation, from which we derive three equations (a system of Riccati equations) which were obtained and studied for parabolic case in [3]. We remark that for the operatorial Riccati equation , using the theory of semigroups, according Theorem 4 and Theorem 5, we find under the abstract form that it has unique solution and this is just the classical solution, which was got in [3].

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