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HAAR SYSTEMS ON A GROUPOID WHOSE ORBIT SPACE IS COUNTABLY SEPARATED

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Abstract. We shall prove a decomposion property of a Haar system and we shall study the structure of the C^*-algebra associated to a locally compact groupoid whose orbit space is countably separated.

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1.Introduction

The construction of the C^*-algebra of a groupoid extends the well-known case of a group. The space of continuous functions with compact support on groupoid is made into a $*$-algebra and endowed with the smallest C^*-norm making its representations continuous. If G is a transitive locally compact second countable groupoid it is well known that the C^*-algebra of G is isomorphic to $C^*(H) \otimes \mathcal{K}(L^2(\mu))$, where H is the isotropy group G_u^u at any unit $u \in G^{(0)}$, μ is an essentially unique measure on $G^{(0)}$, $C^*(H)$ denotes the group C^*-algebra of H, and $\mathcal{K}(L^2(\mu))$ denotes the compact operators on $L^2(\mu)$ (see [5]). In [3] we have obtained a description of the C^*-algebra of a locally compact second countable groupoid G, whose orbit space is Hausdorff, in term of C^*-algebras of the transitive component of G. For obtaining that description we needed some results on the fine structure of the Haar systems.

In this paper we shall see that the decomposition property of a Haar system used in [3] it is true for more general groupoids. In particular, we shall prove that it is true for groupoids whose orbit space is countably separated.

We shall use the notation and terminology of [9], [4], [6].

2.The decomposition of a Haar system

We are exclusively concerned with topological groupoids which are second countable, locally compact Hausdorff. The Borel sets of a topological space are taken to be the σ-algebra generated by the open sets. For quotient of a topological space, we always take the quotient topology, consisting of the sets whose inverse images in the original space are open. The quotient Borel structure is also defined to be the collection of sets whose inverse images in the original space are Borel sets. The quotient Borel structure need not be generated by the quotient topology.

Let G be locally compact second countable groupoid. We shall assume that G admits a continuous Haar system i.e. a family of positive Radon measures on G, $\{\nu^u, u \in G^{(0)}\}$, such that

1) For all $u \in G^{(0)}$, $supp(\nu^u) = G^u$.
2) For all $f : G \to \mathbf{C}$ continuous with compact support,

$$u \to \int f(x)\, d\nu^u(x) \quad \left[: G^{(0)} \to \mathbf{C}\right]$$

is continuous.

3) For all $f : G \to \mathbf{C}$ continuous with compact support, and all $x \in G$,

$$\int f(y)\, d\nu^{r(x)}(y) = \int f(xy)\, d\nu^{d(x)}(y)$$

As a consequence of the existence of continuous Haar systems, $r, d : G \to G^{(0)}$ are open maps.

If μ is a measure on $G^{(0)}$, then the measure $\nu = \int \nu^u d\mu(u)$, defined by

$$\int f(y)\, d\nu(y) = \int \left(\int f(y)\, d\nu^u(y) \right) d\mu(u), \quad f \geq 0 \text{ Borel}$$

is called the measure on G induced by μ. The image of ν by the inverse map $x \to x^{-1}$ is denoted ν^{-1}. μ is said quasi-invariant if its induced measure ν is equivalent to its inverse ν^{-1}. A measure belongings to the class of a quasi-invariant measure is also quasi-invariant. We say that the class is invariant.

If μ is a quasi-invariant measure on $G^{(0)}$ and ν is the measure induced on G, then the Radon-Nikodym derivative $\Delta = \frac{d\nu}{d\nu^{-1}}$ is called the modular function of μ.

Let $G^{(0)}/G$ be the orbit space.($u \sim v \iff \exists\, x \in G$ such that $r(x) = u$ and $d(x) = v$) Let $\pi : G^{(0)} \to G^{(0)}/G$ be the canonical projection, and let $[u] = \{v : v \sim u\}$, be the orbit of u.

In what follows we shall assume that there is a family of positive Radon measures on $G^{(0)}$, $\{\mu^1_{\dot u}, \dot u \in G^{(0)}/G\}$, such that:

(i) For all $\dot u \in G^{(0)}/G$, $\text{supp}(\mu^1_{\dot u}) = [u]$.
(ii) For all f in $C_c(G^{(0)})$,

$$u \to \int f(v)\, d\mu^1_{\pi(u)}(v) \quad \left[: G^{(0)} \to \mathbb{R}\right]$$

is a universally measurable bounded function.

If the orbit space is Hausdorff then there is a family of positive Radon measures satisfying the condition (i) and (ii). (Theorem 3.3 [2]). In the next section we shall see that there is such a family also in the countably separated case. Using a similar argument as in Section 2 [3], we can prove that for each orbit $[u]$, there is a probability $\eta_{\pi(u)}$ supported on $[u]$ such that:

1) $d_*(\nu^v) \sim \eta_{\pi(u)}$ for all $v \in [u]$

2) $\nu^v = \int \nu_{v,w} d\eta_{\pi(u)}(w)$ for all $v \in [u]$, where $\{\nu_{u,v},\ u, v \in G^{(0)},\ u \sim v\}$ is a system of measures with the properties in the preceding section.

3) For all f in $C_c(G^{(0)})$, $u \to \int f(v)\, d\eta_{\pi(u)}(v)$ $[: G^{(0)} \to \mathbb{R}]$ is a universally measurable bounded function.

From this will easily follows that every quasi-invariant measure for $\{\nu^u, u \in G^{(0)}\}$ is equivalent with $\int \eta_{\dot{u}} d\tilde{\mu}(\dot{u})$, for some probability measure $\tilde{\mu}$ on $G^{(0)}/G$.

Proposition 1. Let μ be a quasi-invariant measure on $G^{(0)}$. Let $\mu_1 = \int \eta_{\pi(u)} d\mu(u)$ and ν be the measure induced on G by μ_1. Let $\Delta = \frac{d\nu}{d\nu^{-1}}$ be the modular function of μ_1. There exists a μ_1-conull saturated σ-compact subset Z of $G^{(0)}$and a system of measures,$\{\nu_{u,v},\ u, v \in Z,\ u \sim v\}$, with the following properties:

1)$\nu_{u,v}$ is supported onG_v^u, and $\nu_{u,v} \neq 0$, for all $u, v \in Z, u \sim v$.

2) For all $f \geq 0$ Borel on $G_0 = G|Z$,

$$\int f(y)\, d\nu(y) = \int\int f(y)\, d\nu_{u,v}(y)\, d\eta(u,v) =$$
$$= \int\int\int\int f(y)\, d\nu_{u,v}(y)\, d\eta_{\pi(w)}(v)\, d\eta_{\pi(w)}(u)\, d\mu(w). \quad (1)$$

3) For all $f \geq 0$ Borel on $G_0 = G|Z$,

$$(u, v) \mapsto \int f(y)\, d\nu_{u,v}(y) \ [: (r,d)(G_0) \to \overline{\mathbb{R}}]$$

is an extended real-valued Borel function.

4) For all $f \geq 0$ Borel on $G_0 = G|Z$,

$$\int f(xy)\, d\nu_{d(x),v}(y) = \int f(y)\, d\nu_{r(x),v}(y) \quad \text{for all } x \in G_0,\ v \in [d(x)]$$

Proposition 1. 5) *For all $f \geq 0$ Borel on $G_0 = G|Z$,*

$$\Delta(x) \int f(yx)\, d\nu_{u,r(x)}(y) = \int f(y)\, d\nu_{u,d(x)}(y) \quad \text{for all } x \in G_0,\ u \in [d(x)]$$

6) $\Delta : G_0 \to \mathbb{R}_+^*$ *is a strict homomorphisms.*

7) *For all $f \geq 0$ Borel on $G_0 = G|Z$,*

$$\int f(y)\, d\nu_{u,v}(y) = \int f(y^{-1})\, \Delta(y^{-1})\, d\nu_{v,u}(y) \quad \text{for all } (u,v) \in (r,d)(G_0)$$

8) *For μ-a.a. $u \in G^{(0)} = G|Z$,*

$$\nu^u = \int \nu_{u,v} d\eta_{\pi(u)}(v)$$

Proof. The proof is similar to that given for Lemma 4 [3].

Remark 1. With the notation in the preceding lemma, $\Delta|_{G|[u]}$ is the modular function associated to the Haar system $\{\nu^u, u \in G^{(0)}\}$ and the quasi-invariant measure $\eta_{\pi(u)}$ for μ-a.a. u.

3. Groupoids whose orbit space is countably separated

Assume that the orbit space $G^{(0)}/G$ is countably separated. Let $\pi : G^{(0)} \to G^{(0)}/G$ be the canonical projection.

Applying Theorem 3.4.3/p. 77 [1] to the analytic space $G^{(0)}$, countably separated space $G^{(0)}/G$ and the Borel map π of $G^{(0)}$ onto $G^{(0)}/G$, it follows that there is a universally measurable cross section for π. This means that there is a universally measurable map $\sigma : G^{(0)}/G \to G^{(0)}$ such that $\pi(\sigma(\dot{u})) = \dot{u}$ for all $\dot{u} \in G^{(0)}/G$.

In [8] A. Ramsay proves equivalent conditions in which the orbit space is countably separated. Let

$$R = (r, d)(G) = \{(r(x), d(x)),\ x \in G\}$$

be the graph of the equivalence relation induced on $G^{(0)}$. This R is the image of G under the homomorphism (r, d), so it is a σ-compact groupoid. In particular, R is a F_σ subset of $G^{(0)} \times G^{(0)}$. By Theorem 2.1/p. 363 [8], the quotient Borel structure on $G^{(0)}/G$ is countably separated iff the quotient topology on $G^{(0)}/G$ generates the quotient Borel structure iff $G^{(0)}/G$ is standard as Borel space iff the canonical projection $\pi : G^{(0)} \to G^{(0)}/G$ has a Borel section iff each orbit is locally closed iff $G^{(0)}/G$ is a T_0 space. Thus we may consider that the map σ is Borel.

Let $(K_n)_n$ be an increasing sequence of compact sets with $\bigcup_n K_n = G$. For each n, let $f_n : G \to [0, 1]$ be a continuous with compact support function such that $f_n(x) = 1$ for all $x \in K_n$. Let $a_n(u) = \frac{1}{2^n \nu^u(f_n)}$ if $\nu^u(f_n) > 1$, and $a_n(u) = \frac{1}{2^n}$ otherwise. It is not hard to see that $u \to a_n(u)$ is continuous. Let

$$P_u(x) = \sum_n a_n(u) f_n(x) \text{ for all } x \in G$$

Since $|a_n(u) f_n(x)| \leq \frac{1}{2^n}$, it follows that $(u, x) \to \sum_n a_n(u) f_n(x)$ is uniformly convergent and therefore $(u, x) \to P_u(x)$ is continuous. Thus, for all f in $C_c(G)$,

$$u \to \int f(x) P_u(x) d\nu^u(x) \quad \left[: G^{(0)}/G \to \mathbb{R}\right]$$

is continuous with compact support. If we set $M(u) = \int P_u(x) d\nu^u(x)$, then $0 < M(u) < \infty$ and $u \to M(u)$ is continuous. Let α^u define by

$$\alpha^u(f) = \frac{1}{M(u)} \int f(x) P_u(x) d\nu^u(x)$$

for all f continuous with compact support. Then $u \to \alpha^u$ is continuous and consequently, $u \mapsto d_*(\alpha^u)$is continuous. Let $\eta_u^1 = d_*(\alpha^u)$. Then $u \to \eta_u^1$ is continuous. Let $\eta_{\dot{u}} = \eta^1_{\sigma(\pi(u))}$. For all $f \geq 0$ Borel on $G^{(0)}$ the map

$$u \to \int f(v)\, d\eta_{\pi(u)}$$

is universally measurable and bounded.

Because the hypothesis in the preceding section is satisfied, it follows that if the quotient Borel structure on $G^{(0)}/G$ is countably separated, then the Haar system has the structure described in Proposition 1.

If the quotient Borel structure on $G^{(0)}/G$ is countably separated, then each orbit is locally closed. Thus $G|[u]$ is a locally compact transitive groupoid.

Let $C^*(G)$ be the C^*-algebra of G with respect to the Haar system $\{\nu^u, u \in G^{(0)}\}$ and $\|\|$ be the C^*-norm. For each orbit $[u]$, let $C^*(G|[u])$ be the C^*-algebra of the locally compact groupoid $G|[u]$ with respect to the Haar system $\{\nu^v, v \in [u]\}$ and $\|\|_{\pi(u)}$ be the C^*-norm of this algebra.

For the family of C^*-algebras $\{C^*(G|[u]), \pi(u) \in G^{(0)}/G\}$ we can define the direct sum

$$\sum_{[u]} C^*(G|[u])$$

as the set of all functions

$$\dot{u} \to x_{\dot{u}}\Big(\dot{u} \in G^{(0)}/G,\ x_{\dot{u}} \in C^*(G|[u])\Big)$$

with the property that for every $\varepsilon > 0$ there is a finite subset $E \subset G^{(0)}/G$ such that

$$\|x_{\dot{u}}\|_{\pi(u)} < \varepsilon \quad \text{for } \dot{u} \notin E.$$

The direct product $\prod_{[u]} C^*(G|[u])$has a similar definition except that one takes for its elements all functions $\dot{u} \to x_{\dot{u}}$ satisfying $\sup \|x_{\dot{u}}\|_{\pi(u)} < \infty$. We can make $\sum_{[u]} C^*(G|[u])$ and$\prod_{[u]} C^*(G|[u])$ into C^*-algebras by giving them the "pointwise" operations (for example, $(x_{\dot{u}}) + (y_{\dot{u}}) = (x_{\dot{u}} + y_{\dot{u}})$) and the norm $\|(x_{\dot{u}})_{\dot{u}}\| = \sup \|(x_{\dot{u}})\|_{\pi(u)}$. We consider the following $*$-subalgebra of $\prod_{[u]} C^*(G|[u])$:

$$\Big\{(f_{\pi(u)})_{\pi(u) \in G^{(0)}/G} : (\exists)\ f \in C_c(G) \text{ such that } f|_{G|[u]} = f_{\pi(u)}\Big\}$$

With a similar argument as in the Section 3 of [3] we can prove the following theorem.

Theorem 1. *Let G be a locally compact second countable groupoid equipped with a continuous Haar system. If the quotient Borel structure on $G^{(0)}/G$ is countably separated, then $C^*(G)$ is isomorphic to the completion of*

$$\Big\{(f_{\pi(u)})_{\pi(u) \in G^{(0)}/G} : (\exists)\ f \in C_c(G) \textit{ such that } f|_{G|[u]} = f_{\pi(u)}\Big\}$$

in the norm $\left\|\left(f_{\pi(u)}\right)_{\pi(u)}\right\| = \sup \left\|f_{\pi(u)}\right\|_{\pi(u)}$.

Remark 2. The C^*-algebra of a groupoid G whose orbit space is countably separated can be viewed as a subalgebra of the direct product of C^*-algebras of the transitive component of $G, \prod_{[u]} C^*(G|[u])$. Since $C^*(G|[u])$ is isomorphic to $C^*(G_u^u) \otimes \mathcal{K}\left(L^2\left(\eta_{\pi(u)}\right)\right)$ (see [5]), it follows that $C^*(G)$ can be viewed as a subalgebra of $\prod_{[u]} C^*(G_u^u) \otimes \mathcal{K}\left(L^2\left(\eta_{\pi(u)}\right)\right)$.

Remark 3. If the application

$$r_u : G_u \to G^{(0)}, r_u(x) = r(x)$$

is open for all u, then the orbits $[u]$ of the groupoid G are open and closed subsets of the unit space $G^{(0)}$. It is not hard to see that in this case the C^*-algebra of a groupoid G is isomorphic with the direct sum of C^*-algebras of the transitive component of $G, \sum_{[u]} C^*(G|[u])$. Therefore if the application $r_u : G_u \to G^{(0)}$ is open for all $u \in G^{(0)}$, then $C^*(G)$ does not depend on the Haar system.

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