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SOME QUESTIONS IN THE THEORY OF ŠERSTNEV RANDOM NORMED SPACES

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Abstract. The aim of the present paper is to present a short survey of basic properties of Šerstnev random normed spaces, with emphasis on best approximation problems in such spaces.

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1. Introduction

Probabilistic (or random) metric spaces are spaces on which there is a "distance function" taking as values distribution functions–the "distance" between two points p, q is a distribution function (in the sense of probability theory) $F_{p,q}$, whose value $F_{p,q}(x)$ at $x \in \mathbb{R}$ can be interpreted as the probability that the distance between p and q be less than x. Probabilistic metric spaces were first considered in 1942 by K. Menger [12], who made important contributions to the subject, followed almost immediately by A. Wald [24]. For a good historical account on the development of probabilistic metric spaces see the introductory chapter of the book of B. Schweizer and A. Sklar [17].

In the mean time the theory developed in various directions, an important one being that of fixed points in probabilistic metric spaces. At present, beside Schweizer and Sklar's book mentioned above, there are several books dealing with various aspects of probabilistic metric spaces – V. Istrăţescu [9], I. Istrăţescu and Gh. Constantin [4, 5], V. Radu [16], O. Hadzič [7], O. Hadzič and E. Pap [8].

In 1962 A. N. Šerstnev [18] defined random normed spaces (RNS) as a generalization of usual normed spaces, and studied questions concerning the completeness and the completion of RNS, and the problem of best approximation in RNS. Muštari [13] proved a Mazur-Ulam type theorem: every surjective isometry between two RNS is affine. Some best approximation problems in RNS were studied also by I. Beg [1].

The aim of the present paper is to present a short survey on the basic properties of RNS, with emphasis on best approximation problems.

A *distribution function* is a function $F : \mathbb{R} \to [0,1]$ that is nondecreasing and left continuous on $\mathbb{R}$. We denote by Δ the set of all distribution functions and by D its subclass formed by all $F \in \Delta$ satisfying the conditions:

$$F(-\infty) := \lim_{x \to -\infty} F(x) = 0 \quad \text{and} \quad F(\infty) := \lim_{x \to \infty} F(x) = 1.$$

The *weak convergence* of a sequence (F_n) in Δ to $F \in \Delta$ means that $F_n(x) \to F(x)$ for every continuity point x of the limit function F. There is a metric d_L on Δ, called the modified L ́evy metric, which generates this convergence, and the metric space (Δ, d_L) is compact, hence complete (see [17,Ch. 4]).

In the definition of random normed spaces, Šerstnev [18] used the classes

$$B = \{F : 1 - F \in \Delta\} \quad \text{and} \quad B = \{F : 1 - F \in D\}$$

formed by nonincreasing and left continuous functions from $\mathbb{R}$ to $[0,1]$. For $F \in B$ one requires further that $F(-\infty) = 1$ and $F(\infty) = 0$. The weak convergence is defined as above and $d'_L(F,G) = d_L(1-F, 1-G)$, $F, G \in B$, is a metric on B generating the weak convergence and the metric space (B, d'_L) is compact, hence complete, too. The weak convergence of a sequence (F_n) in B to $F \in B$ is denoted by

The order in B is defined point

2. Random normed spaces

In this section we introduce the random normed spaces following Šerstnev [18, 19, 20, 21, 23]. The main idea is to use functions in the class B^+ as values for the norm instead of positive real numbers and some internal operations acting on B^+, called *triangle functions*, to supply an analogue of the triangle inequality. For this reason, the functions in the class B^+ will be called *distance functions*. A special role play the distance functions defined for $a \geq 0$ by

$$E_a(x) = 1 \ \text{ if } \ x \leq a; \quad E_a = 0 \text{ if } x > 0 \tag{2.1}$$

The function E_0 is the least element of B^+.

A *triangle function* is a mapping $\mu : B^+ \times B^+ \to B^+$ which satisfies the following conditions:

(i) commutativity: $\mu(F, G) = \mu(G, F)$;

(ii) associativity: $\mu(\mu(F,G),H) = \mu(F,\mu(G,H))$;
(iii) $\mu(E_0, F) = F$;
(iv) monotony: $F \le F_1$, and $G \le G_1$ implies $\mu(F,G) \le \mu(F_1,G_1)$;
(v) $\mu(F,G)(x) \le \inf_{t\in[0,1]} \min\{F(tx) + G((1-t)x), 1\}$,
for any $F, G, H \in B^+$ and $x \in \mathbb{R}$.

Two important triangle functions are

$$\mu_b(F,G)(x) = \inf_{t\in[0,1]} \min\{F(tx) + G((1-t)x), 1\}, \tag{2.2}$$

which was used above in (v), and

$$\mu_a(F,G)(x) = \inf_{t\in[0,1]} \max\{F(tx), G((1-t)x)\}. \tag{2.3}$$

A *random normed space* (RNS for short) is a triple (L, ν, μ) where L is a vector space over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, ν is a mapping $\nu : L \to B^+$ and μ is a triangle function satisfying the conditions (i)–(v). The values of ν at $\varphi \in L$ will be denoted by $\nu(\varphi) = \|\varphi\|$ and for $x \in \mathbb{R}$, $\nu(\varphi)(x) = \|\varphi; x\|$.One supposes that the following conditions hold:

(RN1) $\|\varphi\| = E_0 \iff \varphi = \theta$ (the null element of L);
(RN2) $\|a\varphi; x\| = \|\varphi; \frac{x}{|a|}\|$, $\varphi \in L$, $a \in \mathbb{K}$;
(RN3) $\|\varphi + \psi\| \le \mu(\|\varphi\|, \|\psi\|)$, $\varphi, \psi \in L$

We adopt the convention:

$$\|\varphi; \frac{x}{0}\| = E_0(x) = \|0 \cdot \varphi; x\|.$$

The function ν is called a *random norm* on L.

Example 2.1 A usual normed linear space (L, p) can be viewed as a RNS by taking the triangle function μ_a given by (2.3) and putting

$$\|\varphi; x\| = 1 \text{ if } x < p(\varphi); \ \|\varphi; x\| = 0 \text{ if } x > p(\varphi).$$

In fact, we have $\|\varphi\| = E_{p(\varphi)}$, where E_a is given by (2.1).

Example 2.2 A countable normed space is also a RNS.

Let $(L, (p_n))$ be a Hausdorff locally convex space where

$$p_1 \le p_2 \le \dots$$

is a countable family of seminorms generating the topology of L. The fact that the topology of L is Hausdorff means that for every $\varphi \in L$, $\varphi \ne \theta$, there exists $n \in \mathbb{N}$ such that $p_n(\varphi) > 0$. For convenience, we add the null seminorm p_0, $p_0(\varphi) = 0$, $\forall \varphi \in L$, to the sequece of seminorms. Put also $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, where $\mathbb{N} = \{1, 2, ...\}$ is the set of natural nmbers.

Let $(\Omega, , P)$ be a probability space and $\tau : \Omega \to \mathbb{R}$ a random variable on Ω such that

$$\tau(\Omega) \subset \mathbb{N}_0 \quad \text{and} \quad P\{\omega : \tau(\omega) \geq n\} > 0 \quad \forall n \in \mathbb{N}_0. \tag{2.4}$$

Take the triangle function μ_g given by (2.3) and put

$$\|\varphi; x\| = P\{\omega : p_{\tau(\omega)}(\varphi) \geq x\}. \tag{2.5}$$

One can show (see [23]) that (L, ν, μ_g) is an RN space.

Using the random norm one can define a metrizable vector topology on a RNS.

For $0 < \epsilon \leq 1$ and $\delta > 0$ let

$$U_{\epsilon,\delta} = \{\varphi \in L : \|\varphi; \delta\| < \epsilon\}. \tag{2.6}$$

Šerstnev [18, 19, 22] has shown that $\mathcal{U} = \{U_{\epsilon,\delta} : 0 < \epsilon \leq 1, \ \delta > 0\}$ is a local base on L generating a vector topology $\mathcal{T}$ having $\mathcal{U}$ as a base of θ-neighborhoods. Since $\{U_{n^{-1},n^{-1}} ; n \in \mathbb{N}\}$ is a countable base of θ-neighborhoods,it follows that the topology $\mathcal{T}$ is metrizable, so that it is completely determined by the convergent sequences in $(L, \mathcal{T})$. The random norm is continuous with respect to this topology. If $\mu \leq \mu_g$ then each $U_{\epsilon,\delta}$ is convex so that is a locally convex topology on L.

Recall that a subset Y of a topological vector space X is called bounded if for every neighborhood V of $\theta \in X$ there exists $\lambda > 0$ such that $\lambda Y \subset V$. A subset of a RNS is called *bounded* if it is bounded in the vector topology generated by the local base (2.6).

The following theorem follows immediately from the definitions of θ-neighborhoods in a RNS and of a bounded set.

Theorem 2.1. ([21, 14]) *Let Y be a subset of a RNS (L, ν, μ). The following conditions are equivalent:*

1. The set Y is bounded.

2. $\forall \epsilon$, $0 < \epsilon \leq 1$, there exists $c > 0$ such that $\forall \varphi \in Y$ $\|\varphi; c\| < \epsilon$.

3. There exists $F \in B^+$ such that $\forall \varphi \in Y$ $\|\varphi\| \leq F$.

Šerstnev [18] studied also the problem of completeness of a RNS and the possibility to construct a completion of an incomplete RNS.

We present, following Radu [14, 15], some questions concerning spaces of operators between two RNS.

Let (L_i, ν_i, μ_i), $i = 1, 2$, be two RNS and $T : L_1 \to L_2$ a linear operator. The operator T is called *bounded* if for every $F \in B^+$ there exists $G \in B^+$ such that

$$\forall \varphi \in L \quad \nu_1(\varphi) \le F \quad \Rightarrow \quad \nu_2(T\varphi) \le G. \tag{2.7}$$

By Theorem 2.1. the operator T is bounded iff it sends bounded sets in L_1 onto bounded sets in L_2. Denote by ${}_b(L_1, L_2)$ the space of all bounded linear operators from L_1 to L_2, and by (L_1, L_2) the space of all continuous linear operators from L_1 to L_2. Obviously that every continuous linear operator is bounded. Concerning boundedness and continuity we mention the following result:

Proposition 2.2 *If* $\mu_1 \le \mu_2$ *then* ${}_b(L_1, L_2) = (L_1, L_2)$

Suppose that M is a bounded absorbing subset of L_1 and, for a linear operator $T : L_1 \to L_2$, put

$$\bar{\nu}_M(T)(x) = \sup_{\varphi \in M} \|T\varphi : x\| \quad \text{and} \quad \nu_M(T)(x) = \inf_{t < x} \bar{\nu}_M(T)(t). \tag{2.8}$$

We have

Theorem 2.3 *1. If the operator T is bounded then $\nu_M(T) \in B^+$.*

2. The space $(L_1, L_2), \nu_M, \mu_2)$ is a RNS, and the random topology of the space (L_1, L_2) generated by the random norm ν_M defined by (2.8) is stronger than the topology of pointwise convergence, i.e.

$$T_n \to T \quad \text{in} \quad (L_1, L_2) \quad \Rightarrow \quad \forall \varphi \in L_1 \quad T_n\varphi \to T\varphi.$$

Concerning the completeness of the space (L_1, L_2) we can prove:

Theorem 2.4 *If M is a bounded neighborhood of $\theta \in L_1$ and the RNS L_2 is complete then the space $(L_1, L_2), \nu_M, \mu_2)$ is complete.*

Other results concerning the spaces (L_1, L_2) can be found in Radu [14, 15] and Bocşan [2].

A well known theorem of S. Mazur and S. Ulam [11] (see, e.g., Day [6, p.142]) asserts that every surjective isometry between two real normed spaces is an affine mapping.

D. Muštari [13] proved that a similar result holds in the case of RNS. An isometry between two RNS (L_i, ν_i, μ_i), $i = 1, 2$, is an application $T : L_1 \to L_2$ such that $\nu_2(T\varphi - T\psi) = \nu_1(\varphi - \psi)$ for all $\varphi, \psi \in L_1$.

Theorem 2.5 *If T is a surjective isometry between two RNS (L_i, ν_i, μ_i), $i = 1, 2$, then T is an affine mapping.*

3. Best approximation in RNS

The problem of best approximation was first studied by Šerstnev [20, 23]. Let (L, ν, μ) be a RNS and Y a subset of L. For $\varphi \in L$ put

$$\mathcal{A}_\varphi = \{\|\varphi - \psi\| : \psi \in Y\}.$$

The problem of *best ν-approximation* consists in finding the minimal elements of the set $\mathcal{A}_\varphi$, i.e., those elements $\|\varphi - \psi_0\| \in \mathcal{A}_\varphi$ with $\psi_0 \in Y$ for which there is no $\psi \in Y$ with $\|\varphi - \psi\| < \|\varphi - \psi_0\|$. The elements $\psi \in Y$ for which $\|\varphi - \psi\|$ is a minimal element of $\mathcal{A}_\varphi$ are called *elements of best ν-approximation* of φ by elements in Y. Denote by Min $\mathcal{A}_\varphi$ the set of minimal points of $\mathcal{A}_\varphi$.

As usual, the problems which naturally arise are those of the existence, uniqueness, characterization and algorithms for the best ν-approximation elements. In what follows, we shall be concerned only with existence and uniqueness.

Example 3.1 Consider first the example of the RNS associated to a usual normed space as in Example 2.1. Let (L, p) be a normed space and let (L, ν, μ_a) be the associated RNS, with

$$\|\varphi; x\| = E_{p(\varphi)}(x)$$

for $x \geq 0$, where $E_{p(\varphi)}$ is given by (2.1) and μ_a by (2.3). Let Y be a nonempty subset of L and $\varphi \in L \setminus Y$. Since the set $\mathcal{A}_\varphi = \{E_{p(\varphi-\psi)} : \psi \in Y\}$ is totally ordered, it follows that every minimal element of the set is in fact the minimum (the least element) of this set. The equivalence

$$\|\varphi - \psi_0\| \leq \|\varphi - \psi\| \iff p(\varphi - \psi_0) \leq p(\varphi - \psi),$$

which follows from the definition of the random norm, shows that the problem of best ν-approximation and the problem of usual best approximation are equivalent.

Example 3.2 Consider now the RNS associated to a countably normed space $(L, \mathcal{P})$, where $\mathcal{P}$ is an increasing countable family of seminorms $p_1 \leq p_2 \leq \ldots$ generating a Hausdorff locally convex topology on L. Let $(\Omega, \mathcal{A}, P)$ be a probability space and $\tau : \Omega \to \mathbb{R}$ a random variable satisfying a stronger condition than (2.4), namely:

$$\forall n \in \mathbb{N} \quad P\{\omega : \tau(\omega) = n\} > 0. \tag{3.1}$$

Let Y be a nonempty subset of L and $\varphi \in L \setminus Y$. An element $\psi_0 \in Y$ is called $\mathcal{P}$-minimal for φ in Y if there is no $\psi_1 \in Y$ such that

$$\begin{aligned} &(i)\ \forall n \in \mathbb{N} \quad p_n(\varphi - \psi_1) \leq p_n(\varphi - \psi_0), \quad \text{and} \\ &(ii)\ \exists k \in \mathbb{N} \quad p_k(\varphi - \psi_1) \leq p_k(\varphi - \psi_0). \end{aligned} \tag{0.1}$$

The following result holds:

Theorem 3.1 ([23]) *The element $\psi_0 \in Y$ is a best ν-approximation element of $\varphi \in L \setminus Y$ if and only if it is a $\mathcal{P}$ – minimal element for φ in Y.*

We call the subset Y of the RNS (L, ν, μ) *ν-proximinal* if for every $\varphi \in L$ the set Min $\mathcal{A}_\varphi$ is nonempty.

The following existence theorem is inspired by a result in K"othe [10, p.5(12)].

Theorem 3.2 ([3]) *A locally compact closed convex subset of a RNS (L, ν, μ), with $\mu \leq \mu_a$, is ν-proximinal.*

An immediate consequence of the above theorem is the following corollary:

Corollary 3.3 (Šerstnev [23]) *Any finite dimensional subspace of a RNS (L, ν, μ) is ν-proximinal.*

The uniqueness question requires a careful examination, because we work with minimal elements of the norm set and it is possible to exists minimal elements which are incomparable (i.e there is no a relation order between them).

Let (L, ν, μ) be a RNS, $Y \subset L$ and $\varphi \in L$. We say that Y is a *uniqueness set* for the best ν-approximation if for every $\varphi \in L$ and every minimal element F of $\mathcal{A}_\varphi$ there exists at most one element $\psi \in Y$ with $\|\varphi - \psi\| = F$. The set Y is called *ν-Chebyshevian* if it is ν-proximinal and a uniqueness set.

We introduce now a notion, similar to strict convexity of normed spaces, which will guarantee the uniqueness of best ν-approximation elements. A RNS (L, ν, μ_a), where the triangle function μ_a is given by (2.3), is called *strictly convex* if

$$\|\varphi + \psi\| = \mu_a(\|\varphi\|, \|\psi\|) \quad \Rightarrow \quad \exists \lambda \geq 0 \quad \varphi = \lambda\psi. \tag{3.3}$$

(see [23]).

Theorem 3.4 *Let (L, ν, μ) be a RNS and $\varphi \in L$.*

If $\mu = \mu_a$ and $Y \subset L$ is convex then for every minimal element F of $\mathcal{A}_\varphi$ the set

$$\{\psi \in Y : \|\varphi - \psi\| = F\} \tag{3.4}$$

is convex.

If $\mu(F,G)(x) = \max\{F(x), G(x)\}$, $x \in \mathbb{R}$, $F, G \in B^+$, and the subset Y of L is $\frac{1}{2}$-convex then the set (3.4) is $\frac{1}{2}$-convex too.

The uniqueness result is contained in the following theorem:

Theorem 3.5 1. *Every $\frac{1}{2}$-convex subset of a strictly convex RNS (L, ν, μ_a) is a uniqueness set.*

2. *Every closed locally compact convex subset of a strictly convex RNS is ν-Chebyshevian.*

In particular, every finite dimensional subspace of a strictly convex RNS (L, ν, μ_a) is ν-Chebyshevian.

Šerstnev [23] considered also the following approximation problem. Suppose that Y is a subset of a RNS (L, ν, μ). For $\varphi \in L$ let

$$M(\|\varphi\|) = \int_0^\infty \|\varphi; x\| dx$$

be the *mean value of the random norm* and

$$\sigma^2(\|\varphi\|) = \int_0^\infty [x - M(\|\varphi\|)]^2 d_x(1 - \|\varphi; x\|) = 2\int_0^\infty x\|\varphi; x\| dx + \left(\int_0^\infty \|\varphi; x\| dx\right)^2 \tag{3.5}$$

be the *dispersion* of $\|\varphi\|$.

If $\varphi \in L$ then an element $\psi_0 \in Y$ is called an *element of best σ^2-approximation* for φ in Y if

$$\sigma^2(\|\varphi - \psi_0\|) = \inf\{\sigma^2(\|\varphi - \psi\|) : \psi \in Y\}.$$

Šerstnev [23] has shown that each element of best σ^2-approximation is an element of best ν-approximation.

In the case of an ordinary normed space (L, p) and the associated RNS (L, ν, μ_a) (in the sense of Example 2.1), the problems of best ν-approximation and the problem of best σ^2-approximation are equivalent, and are equivalent to the usual best approximation problem in (L, p).

Concerning the existence of the elements of best σ^2-approximation Šerstnev [23] proved:

Theorem 3.6 *Let (L, ν, μ) be a RNS and φ_i, $0 \le i \le m$, be linearly independent elements in L such that $\int_0^\infty x\|\varphi_i; x\| dx < \infty$, $i = 0, 1, \ldots, m$. Then φ_0 has a best σ^2-approximation element in the m-dimensional subspace Y of L, generated by $\varphi_1, \ldots, \varphi_m$.*

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