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(f, Y)– INDUCED BEST APPROXIMATION WITH RESPECT TO A SUBSPACE

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Abstract. The concept of the (f, Y, M) - induced best approximation is introduced in an abstract space. The existence, the uniqueness of the element of the (f, Y, M) - induced best approximation is studied, together with the structure of the set of this type of elements.

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1. Introduction

Many domains of research deal with collections of objects that apparently have no structure. Then, a method of inducing a known structure on this type of sets is necessary and this is usually done by means of numerical descriptions. Various sets of numbers are used to endow such a collection of objects with a structure, which allows a comparison between them, classification, approximations, etc.

In this paper we describe an approach of the process of inducing the best approximation into an amorphous set, in connection with the corresponding possibility of inducing a convexity structure. The consequences of the induced convexity structure on the induced best approximation process is studied in terms of existence and uniqueness of the induced best approximation element. When the element of the induced best approximation is not unique, then the shape of the set of this type of elements is studied.

2. Induced convexity

The induced convexity was defined in [3] and has applications in extending the theory of the best approximation of an element from an abstract space by an element from a given set in a very general framework.

Let X, Y be two arbitrary sets, let M be a nonempty subset of Y and let us consider the set valued mapping $s : 2^Y \to 2^Y$. Suppose that Y has a convexity structure. A mapping $f : X \to Y$, will be used to endow X with a convexity structure, based on the structure of Y and, when required, s and/or set M.

Let $a, b \in X$. A set $< a, b >_{s,f,M} = f^{-1}(s(\{f(a), f(b)\}) \bigcap M)$ is defined for each pair of points a, b, to replace the straight line segment determined by a and b. Clearly $< a, b >_{s,f,M} = \{z \in X \mid f(z) \in s(\{f(a), f(b)\}) \bigcap M\}$.

Definition 2.1 *Subset A of X is said to be induced seg-convex with respect to s, f and M if $< a, b >_{s,f,M} \subseteq A$, for any $a, b \in A$.*

In what follows we use the notation $G_{seg,s,f,M}$ for the set of all subsets of X which are induced seg-convex with respect to (s, f, M).

Example 2.1

Let $X = \{-1, 1, 2\}$, $Y = \{-1, 0, 1, 2\}$ and let $f : X \to Y$ given by $f(-1) = 1, f(1) = 1, f(2) = 2$.

Let $s : 2^Y \to 2^Y$ given by

$$s(\emptyset) = \emptyset,\ s(\{-1\}) = \{-1\},$$
$$s(\{0\}) = \{0\},\ s(\{1\}) = \{0, 1\},$$
$$s(\{2\}) = \{0, 1, 2\},\ s(\{-1, 0\}) = \{0\},$$
$$s(\{-1, 1\}) = \{0, 1\},\ s(\{-1, 2\}) = \{0, 1, 2\},$$
$$s(\{0, 1\}) = \{0, 1\},\ s(\{0, 2\}) = \{0, 1, 2\},$$
$$s(\{1, 2\}) = \{1, 2\},\ s(\{-1, 0, 1\}) = \{0, 1\},$$
$$s(\{-1, 0, 2\}) = \{0, 2\},\ s(\{-1, 1, 2\}) = \{1, 2\},$$
$$s(\{0, 1, 2\}) = \{0, 1, 2\},\ s(\{-1, 0, 1, 2\}) = \{-1, 0, 1, 2\}.$$

Let $M = Y$ and $A = \{-1, 1\}$. Because

$$s(\{f(-1), f(1)\}) = \{0, 1\} \not\subset \{1\} = f(\{-1, 1\}) = f(A),$$

and

$$< -1, 1 >_{s,f,M} = \{z \in X | f(z) \in \{0, 1\}\} = \{-1, 1\} \subseteq A$$

it follows that set A is seg-convex with respect to s, f and M.

It is well known that any straight-line segment is a convex set and the intersection of two convex sets is also convex. This does not remain valid in the case of the induced convexity. Additional conditions make these classical properties fulfil in the case of the induced convexity.

Theorem 2.1 *If $f : X \to Y$ and $s : 2^Y \to 2^Y$ are given functions satisfying the conditions that*

$$s(A) \subseteq s(B) \text{ for any } A,\ B \in 2^Y \text{ with } A \subseteq B \tag{1}$$

and

$$s(s(B) \bigcap M) \subseteq s(B) \bigcap M, \text{ for each } B \in 2^Y, \tag{2}$$

then for any two points a and b of X the set $< a, b >_{s,f,M}$ is induced seg-convex with respect to (s, f, M).

Proof. Let a, b in X. Let $c, d \in < a, b >_{s,f,M} = f^{-1}(s(\{f(a), f(b)\}) \bigcap M)$.

We have

$$f(c) \in s(\{f(a), f(b)\}) \bigcap M, \text{ and } f(d) \in s(\{f(a), f(b)\}) \bigcap M.$$

It results that $s(\{f(c), f(d)\}) \subseteq s(s(\{f(a), f(b)\}) \bigcap M)$, and, in view of (2), it follows $s(\{f(c), f(d)\}) \bigcap M \subseteq s(\{f(a), f(b)\}) \bigcap M$.

This implies that

$$< c, d >_{s,f,M} = f^{-1}(s(\{f(c), f(d)\}) \bigcap M) \subseteq$$

$$\subseteq f^{-1}(s(\{f(a), f(b)\}) \bigcap M) = < a, b >_{s,f,M}$$

Hence $< a, b >_{s,f,M}$ is induced seg-convex with respect to (s, f, M).

Theorem 2.2 *If X, Y, M are nonempty sets, and $f : X \to Y$, $s : 2^Y \to 2^Y$ are given functions, then set $G_{seg,s,f,M}$ is a convexity space.*

Proof. It is easy to see that sets $\emptyset$ and X are induced seg-convex with respect to (s, f, M). Let us consider the family $(A_j)_{j \in J}$, with $A_j \in G_{seg,s,f,M}$, and let a, b be elements of $\bigcap_{j \in J} A_j$. Then $a, b \in A_j$, for all $j \in J$. As sets A_j, $j \in J$, are induced seg-convex with respect to (s, f, M), we have $< a, b >_{s,f,M} \subseteq A_j$, for each $j \in J$. Then $< a, b >_{s,f,M} \subseteq \bigcap_{j \in J} A_j$. Therefore $\bigcap_{j \in J} A_j$ is induced seg-convex with respect to (s, f, M).

The relationship between the induced seg-convexity with respect to (s, f, M) and the classical convexity is established, in specific conditions. Let us suppose that Y is a linear space and M is a linear subspace of Y. Function $s : 2^Y \to 2^Y$ is defined by $s(C) = conv(C)$, for each $C \in 2^Y$. Here, $conv(C)$ means the convex hull of set C in the classical sense. Subset A of X is called induced seg-convex with respect to f and M if A is induced seg-convex with respect to (s, f, M).

Theorem 2.3 *If function $f : X \to Y$ is injective, if set $f(X)$ is classically convex in Y, set A is induced seg-convex with respect to f and M and $f(A) \subset M$, then set $f(A)$ is also classically convex in M.*

Proof. Let be $u', u" \in f(A)$. There are $a', a" \in A$, such that $u' = f(a')$, and $u" = f(a")$. Let $t \in]0, 1[$. Because set $f(X)$ is convex, we have $u = tu' + (1-t)u" \in f(X)$. From the induced convexity of A, it follows that $f^{-1}(u) \in < a', a" > \subseteq A$. Then there is $a \in A$, such that $f(a) = u$. We get that

$$u = tf(a') + (1-t)f(a") \in f(A) \subseteq M.$$

Because t is arbitrarily chosen in $]0, 1[$, it results that $f(A)$ is a convex set in M.

3. The element of the (f, Y, M) - induced best approximation

In this paragraph we shall suppose that X is an arbitrary set, $(Y, +, \cdot, || \ \ ||)$ an H-normed linear space and $f : X \to Y$ an injective function. Set M is assumed to be a subspace of Y. Function $s : 2^Y \to 2^Y$ is given by $s(C) = conv(C)$, for all $C \in 2^Y$. As above, subset A of X is called induced seg-convex with respect to f and M, if A is induced seg-convex with respect to s, f, and M. Obviously, set $A \subseteq X$ is seg-convex with respect to f and M if and only if for each pair of points $a, b \in A$ we have

$$< a, b >= \{x \in X | f(x) \in \{(1-t) \cdot f(a) + t \cdot f(b) | t \in [0,1]\} \bigcap M\} \subseteq A.$$

Then, it is easy to see that $a \in < a, b >$, and $b \in < a, b >$, for any a, $b \in X$.

Let $x^0 \in X$ and $A \subseteq X$, $A \neq \emptyset$.

Definition 3.1 *A point $a^0 \in A$ is said to be an element of the (f, Y, M) - induced best approximation of x^0 by elements of A if*

$$||f(a^0) - f(x^0)|| \leq ||f(a) - f(x^0)||, \tag{3}$$

for all $a \in A$ with $f(a) \in M$.

We recall that in the classical case, i.e. $X = \mathbf{R}^n$, if $A \subseteq \mathbf{R}^n$ is a convex set and $y^0 \in \mathbf{R}^n$, then there is at most one element of the best approximation (in classical sense) of y^0 by elements of A. In what follows, we shall show that it is possible to prove a similar property under additional hypothesis for an induced seg-convex set with respect to f and M.

Theorem 3.1 *If x^0 is a given point of X, and A is a nonempty induced seg-convex set with respect to f and M such that $f(A)$ is a classically convex set in M, then there is at most one element $a^0 \in A$ of the (f, Y, M) - induced best approximation of x^0 by elements of A.*

Proof. Two cases may arise:

Case 1: $x^0 \in A$. Then, from the injectivity of f and from $f(A) \subseteq M$, it follows that x^0 is the unique (f, Y, M) - induced best approximation point of x^0, by elements of A.

Case 2: $x^0 \notin A$. We suppose that there are at least two elements a^0 and a of the (f, Y, M) - induced best approximation of x^0, by elements of A. Then we have

$$||f(a) - f(x^0)|| = ||f(a^0) - f(x^0)|| \neq 0. \tag{4}$$

Because f is an injective function we have $f(a) \neq f(a^0)$. Since $f(a)$, $f(a^0) \in M$, then

$$\{f(a), f(a^0)\} \subset < f(a), f(a^0) > \subseteq f(A) \subseteq M. \tag{5}$$

It follows that there is $\gamma \in f(A) \setminus \{f(a), f(a^0)\}$, such that

$$\gamma = \frac{1}{2} f(a) + \frac{1}{2} f(a^0). \tag{6}$$

Since the set A is induced seg-convex with respect to f and M, we get

$$f^{-1}(\gamma) \in A. \tag{7}$$

Then there is $c \in A$, such that

$$\gamma = f(c) = \frac{1}{2}f(a^0) + \frac{1}{2}f(a). \tag{8}$$

From (6) and (7) we have

$$\begin{aligned} ||f(c) - f(x^0)||^2 &= \tfrac{1}{4}||(f(a^0) + f(x^0)) + (f(a) - f(x^0))||^2 < \\ &< \tfrac{1}{4}(f(a^0) - f(x^0)) + (f(a) - f(x^0))||^2 + \\ &+ ||(f(a^0) - f(x^0)) + (f(a) - f(x^0))||^2 \end{aligned}$$

Applying the parallelogram equality for the H-norm we get

$$||(f(a^0) - f(x^0)) + (f(a) - f(x^0))||^2 + ||(f(a^0) - f(x^0)) - (f(a) - f(x^0))||^2 =$$

$$= 2||f(a^0) - f(x^0)||^2 + 2||f(a) - f(x^0)||^2.$$

Then $||f(c) - f(x^0)|| < ||f(a^0) - f(x^0)||$. This contradicts the fact that a^0 is an element of the (f, Y, M)-induced best approximation of x^0, by elements of A.

Remark 3.1 If $f(A)$ is not a classically convex set in M, then the conclusion of theorem 4 does not remain true.

Example 3.1. Let be

$X = \{(0,1), (0,0), (1,0)\}, Y = M = \mathbf{R}^2, x^0 = (0,0), A = \{(0,1), (1,0)\}$.

Taking $f : X \to Y$, $f(x_1, x_2) = (x_1, x_2)$, for each $(x_1, x_2) \in X$, then $f(A) = A$, which is not classically convex in M. It is easy to see that set A is induced seg-convex with respect to f and M. But points $(0,1)$, and $(1,0)$, are simultaneously elements of the (f, Y, M) - induced best approximation of x^0 by elements of A.

From theorem 4 and theorem 1 we get:

Corollary 3.1 *Let us suppose that*

i) Function $f : X \to Y$ is injective,

ii) $f(X)$ is a convex set,

iii) x^0 is a given point of X,

iv) A is a nonempty induced seg-convex set with respect to f and M,

v) $f(A) \subseteq M$.

Then there is at most one element of the (f, Y, M) - induced best approximation of x^0 by elements of A.

Theorem 3.2 *Let us suppose that:*

i) Function $f : X \to Y$ is injective,

ii) x^0 is a given point of X,

iii) A is a nonempty induced seg-convex set with respect to f and M,

iv) $f(A) \subseteq M$.

Then the set $A(x^0)$ of all elements of the (f, Y, M) - induced best approximation of x^0 by elements of A is also induced seg-convex with respect to f and M.

Proof. Two cases are possible:

Case 1. $\mathrm{card}(A(x^0)) \in \{0,1\}$; obviously $A(x^0)$ is an induced seg-convex set with respect to f and M.

Case 2. $\mathrm{card}(A(x^0)) > 1$. Let $a', a'' \in A(x^0)$, and $\lambda = \|f(x^0) - f(a')\| = \|f(x^0) - f(a'')\|$.

Let be $a \in f^{-1}(< f(a'), f(a'') > \bigcap M)$. Then there is $t \in [0,1]$ such that

$$f(a) = tf(a') + (1-t)f(a'').$$

We have

$$\|f(x^0) - f(a)\| = \|t(f(x^0) - f(a')) + (1-t)(f(x^0) - f(a''))\| \leq$$

$$\leq t\|f(x^0) - f(a')\| + (1-t)\|f(x^0) - f(a'')\| = \lambda.$$

On the other hand, we have

$$\|f(x^0) - f(a)\| \geq \lambda.$$

It follows that $\|f(x^0) - f(a)\| = \lambda$. Then $a \in A(x^0)$. Hence set $A(x^0)$ is induced seg-convex with respect to f and M.

Particular cases of theorems 4 and 6 can be found in [1] and [4]. It would be of interest to extend these results in convexity structures that are not necessarily of segmental type, according to the terminology from [1], but more general ones as in [1] or [2].

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