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A GENERALIZATION OF TATE ALGEBRAS IN ONE INDETERMINATE OVER LOCAL NON-ARCHIMEDEAN FIELDS

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Abstract. The Tate algebras T_n over a complete non-archimedean field K, consisting of all power series in n variables converging on the "closed" unit polydisc in K^n, lead to the definition of affinoid algebras ([2], Part B or [3], Ch.2). In this paper, by using particular series of functions converging on the closed unit disc in K, we extend the algebra T_1 over a local non-Archimedean field.

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Let K be a commutative field and let $x_1, x_2, \ldots, x_m$ be distinct elements of K. If n is a nonnegative integer we consider the quotient $q(n)$ and the remainder $r(n)$ obtained when n is divided by m. We construct the polynomials

$$u_n(X) = \prod_{k=1}^{m} (X - x_k)^{q(n)+\sigma(r(n)-k)}, \; n = 0, 1, 2, \ldots \tag{1}$$

where $\sigma(x)$ is equal to 0 if $x < 0$ and 1 otherwise. If $P(X) \in K[X]$, we denote by $P_t(z_1; z_2; \ldots; z_t)$ the divided difference of $P(X)$ with respect to t distinct elements $z_1, z_2, \ldots, z_t \in K$.

Lemma 1. *Let K be a commutative field and let $x_1, x_2, \ldots, x_m$ be distinct elements of K. Then there exist the elements $p_k(s,t) \in K$, where $s, t \in \{0, 1, \ldots, m-1\}$, $\max\{s,t\} \le k \le s+t$, such that in the ring $K[X]$, for all nonnegative integers i and j,*

$$u_i(X)u_j(X) = \sum_{k=\max(r(i),r(j))}^{r(i)+r(j)} p_k(r(i), r(j)) u_{(q(i)+q(j))m+k}(X), \tag{2}$$

where $p_k(s,t) = u_{\min\{s,t\}; k-\max\{s,t\}+1}(x_{\max\{s,t\}+1}; x_{\max\{s,t\}+2}; \ldots; x_{k+1})$, $u_i(X)$ is given by (1), $x_k = x_{r(k)}$, if $r(k) \neq 0$ and $x_k = x_m$, if $r(k) = 0$.

Proof. If $r(j) = 0$, then $j = q(j)m$ and by (1) $u_i(X)u_j(X) = u_{i+j}(X)$. Hence it follows (2) with $p_i(0,t) = 1$. Now we suppose that $r(j) \neq 0$. Then $u_i(X)u_j(X) = u_{(q(i)+q(j))m}(X)u_{r(i)}(X)u_{r(j)}(X)$. We suppose $r(i) \ge r(j)$. Then, by Newton interpolation formula with respect to $x_{r(i)+1}; x_{r(i)+2}; \ldots; x_{r(i)+r(j)+1}$, we obtain that $u_{r(j)}(X) =$

$\sum_{v=0}^{r(j)} \left(u_{r(j);v+1}\left(x_{r(i)+1}; x_{r(i)+2}; \ldots; x_{r(i)+v+1}\right) \prod_{w=1}^{v} \left(X - x_{r(i)+w}\right)\right)$. Hence it follows (2).
$\square$

We consider the set $S_K(x_1, \ldots, x_m)$ of formal series of the form $f = \sum_{i=0}^{\infty} a_i u_i(X)$, where $a_i \in K$ and $u_i(X)$ are given by (1). If $f, g = \sum_{i=0}^{\infty} b_i u_i(X) \in S_K(x_1, \ldots, x_m)$, we define addition and multiplication of f and g as follows:

$$f + g = \sum_{i=0}^{\infty} (a_i + b_i) u_i(X), \tag{3}$$

$$fg = \sum_{i=0}^{\infty} c_i u_i(X), \text{ where } c_i = \sum_{\substack{\max\{s+q(t)m,\, t+q(s)m\} \le i \\ s+t \ge i}} p_{i-(q(s)+q(t))m}(r(s), r(t)) a_s b_t \tag{4}$$

and $p_i(s,t)$ are defined in Lemma 1. Consider $f = \sum_{i=0}^{\infty} a_i u_i(X) \in S_K(x_1, \ldots, x_m)$ a nonzero series. The smallest index i for which a_i is different from zero will be called *the order* of f and will denoted $o(f)$.

Lemma 2. *There exists an injective map $\varphi : K[X] \to S_K(x_1, \ldots, x_m)$ such that for all $P(X), Q(X) \in K[X]$, $\varphi(P(X) + Q(X)) = \varphi(P(X)) + \varphi(Q(X))$ and $\varphi(P(X)Q(X)) = \varphi(P(X))\varphi(Q(X))$, where the addition and the multiplication in $S_K(x_1, \ldots, x_m)$ are defined by* (3) *and* (4).

Proof. For $P(X) \in K[X]$ we define $\varphi(P(X))$ by induction on the degree of $P(X)$. If $\deg(P(X)) \le m - 1$, by means of Newton interpolation formula, we put $\varphi(P(X)) = \sum_{k=1}^{m} P_k(x_1; x_2; \ldots; x_k)(x - x_1) \ldots (x - x_{k-1})$. If $\deg(P(X)) \ge m$, there exist $Q(X), R(X) \in K[X]$, with $\deg(R(X)) < m$, such that $P(X) = Q(X)u_m(X) + R(X)$. Then we define $\varphi(P(X)) = \varphi(Q(X))u_m(X) + \varphi(R(X))$. Because a system of polynomials which have different degrees is linearly independent over K, it follows easily the lemma. $\square$

Theorem 1. *If the addition and the multiplication are defined by* (3) *and* (4), *then the set $S_K(x_1, \ldots, x_m)$ becomes a commutative K-algebra which is a principal ideal domain.*

Proof. Since, for every nonnegative integers s and t, the system of polynomials $u_{(q(s)+q(t))m}(X), u_{(q(s)+q(t))m+1}(X), \ldots, u_{s+t}(X) \in K[X]$ is linearly independent over K, by (2) it follows that for each i, $p_i(s,t) = p_i(t,s)$. If we denote $\varphi(K[X])$ also by $K[X]$, where φ is given in Lemma 2, we obtain that $K[X]$ becomes a commutative K-algebra which is an integral domain, if the addition and the multiplication are defined by (3) and (4). If we consider $f = \sum_{i=0}^{\infty} a_i u_i(X) \in S_K(x_1, \ldots, x_m)$, we put $f_n = \sum_{i=0}^{n} a_i u_i(X)$. Then each coefficient of $f + g$, fg from (3) and (4) can be obtained as a sum or a multiplication of polynomials of the form f_n and g_n. Hence the statement that $S_K(x_1, \ldots, x_m)$ is an integral domain can be reduced to $K[X]$, whence it follows that $S_K(x_1, \ldots, x_m)$ is an integral domain. To show that $S_K(x_1,\dots,x_m)$ is a principal ideal domain we present a proof similar to the one given in [4], p.138 in the case of formal power series. Let $\mathcal{I}$ be an ideal in $S_K(x_1,\dots,x_m)$ and $f=\sum_{i=o(f)}^{\infty} a_i u_i(X)\in\mathcal{I}$, where $o(f)$ is the smallest order $o(g)$ for all $g\in\mathcal{I}$. We first show that we can choose f such that, for every $g\in\mathcal{I}$, there exists $v(X)\in K[X]$ satisfying the condition

$$o(g-v(X)f)>o(g). \tag{5}$$

Because $u_m(X)u_i(X)=u_{m+i}(X)$, for every i, it is enough to consider g such that $o(g)\in\{o(f)+1,o(f)+2,\dots,o(f)+m-1\}$. Assume the contrary. That means that there exists $g=\sum_{i=o(g)}^{\infty} b_i u_i(X)\in\mathcal{I}$, with the smallest index $o(g)\in\{o(f)+1,o(f)+2,\dots,o(f)+m-1\}$, such that for every $P(X)\in K[X]$, $o(g-P(X)f)\le o(g)$. We put $t=o(g)-o(f)$.

$$d_t(f)=\sum_{k=0}^{t}\frac{a_{o(f)+k}}{\prod_{j=o(f)+k+1}^{o(g)}(x_{o(f)+t+1}-x_j)}, \tag{6}$$

$$v(X)=\alpha\prod_{j=o(f)+1}^{o(g)}(X-x_j)\text{, where} \tag{7}$$

$$\alpha=\begin{cases}\frac{b_{o(g)}}{d_t(f)}, & \text{if } d_t(f)\neq 0\\ \prod_{j=o(f)+1}^{o(g)}(x_{o(f)+t+1}-x_j)^{-1}, & \text{if } d_t(f)=0\end{cases} \tag{8}$$

Hence $v(X)f=\alpha\sum_{i=o(f)} a_i u_{o(g)}(X)\prod_{j=o(f)+1}^{i}(X-x_j)$. Since by Newton interpolation formula $\prod_{j=o(f)+1}^{i}(X-x_j)=\prod_{j=o(f)+1}^{i}(x_{o(g)+1}-x_j)+(X-x_{o(g)+1})w(X)$, for $i\le o(f)+m-1$, where $w(X)\in K[X]$, we obtain

$$v(X)f=v(x_{o(g)+1})d_t(f)u_{o(g)}(X)+f_1\text{, with } o(f_1)>o(g). \tag{9}$$

If $d_t(f)\neq 0$, by (8) and (9) it follows (5). We suppose now that $d_t(f)=0$. If we put $\bar f=f+g$, since $g-v(X)\bar f=(b_{o(g)}-v(x_{o(g)+1})d_t(f)-v(x_{o(g)+1})b_{o(g)})u_{o(g)}(X)+f_2=f_2$, where $o(f_2)>o(g)$, we obtain that $o(g-v(X)\bar f)>o(g)$. But $o(\bar f)=o(f)$ and for every $\overline{g}$ with $o(\overline{g})<o(g)$, $o(\overline{g}-v_1(X)f)>o(\overline{g})$ implies $o(\overline{g}-v_1(X)\bar f)=o(\overline{g}-v_1(X)f-v_1(X)g)\ge\inf\{o(\overline{g}-v_1(X)f),o(v_1(X)g)\}>o(\overline{g})$. Now we can replace f by $\bar f$ and (5) follows by induction on $o(g)$.

Let $g_1 = \sum_{i=o(g_1)}^{\infty} b_{i,1}u_i(X)$ be an element of $\mathcal{I}$. Then we may consider $g_1 \neq 0$. By (5) there exists $v_1(X) \in K[X]$ given by (7) such that $g_2 = g_1 - b_{i,1}v_1(X)f \in \mathcal{I}$, and $o(g_2) > o(g_1)$. Then $g_1 = g_2 + b_{i,1}v_1(X)f$. By successive application of this method, for every n, we obtain

$$g_1 = h_n(X)f + g_{n+1}, \tag{10}$$

where $h_n(X) = \sum_{k=1}^{n} b_{i,k}v_k(X) \in K[X]$, $g_{n+1} = \sum_{i=o(g_{n+1})}^{\infty} b_{i,n+1}u_i(X) \in \mathcal{I}$, $\lim_{k\to\infty} \deg v_k(X) = \infty$. Hence, by Newton interpolation formula and Lemma 2, it follows that we can construct $h \in S_K(x_1, ..., x_m)$ such that for every s there exists h_n such that $h - h_n = \sum_{i=s}^{\infty} c_i u_i(X)$. Thus, by (10) it follows that $g_1 = hf$ and $S_K(x_1, ..., x_m)$ is a principal ideal domain. □

Corollary. *If* $f = \sum_{i=0}^{\infty} a_i u_i(X) \in S_K(x_1, ..., x_m)$, *then* f *is a unit in* $S_K(x_1, ..., x_m)$ *if and only if for every* $j \in \{0, 1, ..., m-1\}$

$$C_j = \sum_{v=0}^{j} p_j(v, j)a_v \tag{11}$$

are non-zero elements in K.

Proof. Since f is a unit in $S_K(x_1, ..., x_m)$ if and only if there exists $g = \sum_{i=0}^{\infty} b_i u_i(X) \in S_K(x_1, ..., x_m)$ such that $c_0 = 1$ and $c_i = 0$, for all $i \geq 1$, where c_i are given by (4), it follows that

$$\sum_{\substack{\max\{s+q(t)m,\, t+q(s)m\} \leq i \\ s+t \geq i}} p_{i-(q(s)+q(t))m}(r(s), r(t))a_s b_t = \delta_{i,0}, \tag{12}$$

where $\delta_{i,j}$ is the Kronecker delta symbol. But, for a fixed i, the corresponding equation of (12) contains the elements b_j with $j \leq i$ and the coefficient of b_i is $C_{r(i)}$ given by (11). Then, if $i_1 \equiv i_2 (\mathrm{mod}\, m)$, the coefficients of b_{i_1} and b_{i_2} in the corresponding equations of (12) for $i = i_1$ and $i = i_2$ respectively, are the same. Hence the system (12) has a unique solution b_i, $i = 0, 1, ...,$ if and only if $C_j \neq 0$ for every $j \in \{0, 1, ..., m-1\}$, whence it follows the corollary. □

Now we consider K a commutative field with a non-trivial non-archimedean valuation $|\,|$. If K is a locally compact field, then it is called a *local field*. If A is a commutative ring with identity and $\|\,\|$ is a non-archimedean norm on A, we consider the sets: $\overset{\circ}{A} = \{x \in A; \|x\| \leq 1\}$ and $\overset{\vee}{A} = \{x \in A; \|x\| < 1\}$ (see [2], Chapter 1). Then $\overset{\circ}{A}$ is a commutative ring with identity and $\overset{\vee}{A}$ is an ideal in $\overset{\circ}{A}$. We denote the residue ring $\overset{\circ}{A} / \overset{\vee}{A}$ by $\bar{A}$. Let

$A = K$ be a commutative field with a non-trivial non-archimedean valuation $||$. Then $\overset{\circ}{K}$ is a local ring called *the valuation ring* of $||$ and $\overset{\vee}{K}$ is the maximal ideal of $\overset{\circ}{K}$. If K is a local field, then it is a complete field and the residue field $\widetilde{K}$ is a finite field of order $m = p^s$, where p is the characteristic of $\widetilde{K}$ ([1], Proposition 2.3.3, p. 51). Let $x_1, x_2, ..., x_m$ be elements of $\overset{\circ}{K}$ such that the cosets $x_j + \overset{\vee}{K}$, $j \in \{1, 2, ..., m\}$, are distinct. We denote

$$TS_K(x_1, ..., x_m) = \{f \in S_K(x_1, ..., x_m) ; \exists M > 0, |a_i| < M, \text{ for all } i\}. \tag{13}$$

If $f = \sum_{i=0}^{\infty} a_i u_i(X) \in TS_K(x_1, ..., x_m)$, the real number

$$\|f\| = \sup_i |a_i| \tag{14}$$

is well defined. As usual we call $\| \|$ *the Gauss norm on* $TS_K(x_1, ..., x_m)$.

Theorem 2. *$TS_K(x_1, ..., x_m)$ is a subalgebra of the K- algebra $S_K(x_1, ..., x_m)$ and the Gauss norm given by (14) is a K-algebra non-archimedean norm on $TS_K(x_1, ..., x_m)$ making it into a K-Banach algebra.*

Proof. It easily to see that $TS_K(x_1, ..., x_m)$ is a subalgebra of the K-algebra $S_K(x_1, ..., x_m)$ and the Gauss norm given by (14) is a K-algebra norm on $TS_K(x_1, ..., x_m)$. Now we show that $(TS_K(x_1, ..., x_m), \| \|)$ is complete. Let $f^{[n]} = \sum_{i=0}^{\infty} a_{i,n} u_i(X)$, $n = 1, 2, ...$ be a Cauchy sequence in $TS_K(x_1, ..., x_m)$. Since

$$|a_{i,n+1} - a_{i,n}| \le \left\| f^{[n+1]} - f^{[n]} \right\|, \tag{15}$$

for a fixed i, each sequence $a_{i,n}$, $n = 0, 1, 2, ...$ is a Cauchy sequence in K. For $i = 0, 1, 2, ...$, let $a_i \in K$ be the limit of this sequence. Set $f = \sum_{i=0}^{\infty} a_i u_i(X) \in S_K(x_1, ..., x_m)$. Then it follows easily that $f \in TS_K(x_1, ..., x_m)$ and $\lim_{n\to\infty} \|f - f^{[n]}\| = 0$. $\square$

Proposition 1. *The Gauss norm is a valuation on $TS_K(x_1, ..., x_m)$ and the K-algebra $\widetilde{TS_K(x_1, ..., x_m)}$ is isomorphic to $S_{\widetilde{K}}(x_1, ..., x_m)$.*

Proof. If $\pi : \overset{\circ}{K} \to \widetilde{K}$ is the canonical map, then we define the morphism $\Phi : TS_K(x_1, ..., x_m) \to S_{\widetilde{K}}(x_1, ..., x_m)$ by setting $\Phi\left(\sum_{i=0}^{\infty} a_i u_i(X)\right) = \sum_{i=0}^{\infty} \pi(a_i) \overline{u}_i(X)$, where $\overline{u}_i(X)$ is the canonical image of $u_i(X)$ in $\widetilde{K}[X]$. Since $||$ is a nonarchimedean valuation there exists i_0 such that $\sup_i |a_i| = a_{i_0}$. Hence $\mathrm{Ker}\Phi$ is $TS_K(x_1, ..., x_m)$ and $TS_K\widetilde{(x_1, ..., x_m)}$ is isomorphic to $S_{\widetilde{K}}(x_1, ..., x_m)$. Now by Theorem 1 and Proposition 1 of [2], p.43, it follows that the Gauss norm is a valuation on $TS_K(x_1, ..., x_m)$. $\square$

Corollary. *A non-zero element* $f = \sum_{i=0}^{\infty} a_i u_i(X) \in TS_K(x_1, ..., x_m)$ *is a unit in* $TS_K(x_1, ..., x_m)$ *if and only if all the elements* $\widetilde{C}_j = \sum_{v=0}^{j} \pi(p_j(v,j) a_v a_{i_0}^{-1})$, *where* $j \in \{0, 1, ..., m-1\}$ *and* $|a_{i_0}| = \|f\|$, *from* $\widetilde{K}$ *are different from zero.*

Proof. Since $\|\|$ is a valuation we may assume $\|f\| = 1$. By Proposition 8, p.30 of [2], f is unit in $TS_K(x_1, ..., x_m)$ if and only if $\tilde{f}$ is a unit in $TS_K\widetilde{(x_1, ..., x_m)}$. Then the corollary follows by Corollary of Theorem 1 and Proposition 1. □

Theorem 3. *The* K*-Banach algebra* $TS_K(x_1, ..., x_m)$ *is a principal ideal domain strictly containing a* K*-Banach subalgebra homeomorphic to the Tate algebra* T_1 *over* K.

Proof. Let $\mathcal{I}$ be an ideal in $TS_K(x_1, ..., x_m)$. By using the notation from the proof of Theorem 1, since $x_1, ..., x_m \in \widetilde{K}$, it follows easily that $h \in TS_K(x_1, ..., x_m)$. Hence $TS_K(x_1, ..., x_m)$ is a principal ideal domain.

We consider the Gauss valuation on the K-algebra $K[X]$, $|a_0 + a_1 X + ... + a_n X^n| = \max_{1 \le i \le n} |a_i|$. Then the map φ given in Lemma 2 is a contraction from $(K[X], |\,|)$ to $(TS_K(x_1, ..., x_m), \|\,\|)$. Because $(K[X], |\,|)$ is a dense K-subalgebra of T_1 we can extend φ from T_1 to $TS_K(x_1, ..., x_m)$. By Open Mapping Theorem of Banach ([2], p.123), it follows that T_1 is homeomorphic to a K-Banach subalgebra of $TS_K(x_1, ..., x_m)$.

To prove that $\varphi(T_1) \neq TS_K(x_1, ..., x_m)$ it is enough to note that $TS_K\widetilde{(x_1, ..., x_m)} \cong S_{\widetilde{K}}(x_1, ..., x_m)$ and $\widetilde{T_1} = \widetilde{K}[X]$. □

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