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NONNEGATIVE SOLUTIONS FOR BOUNDARY VALUE PROBLEM

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Abstract. We establish what conditions require for nonlinear integral equation

$$u(x) = g(x) + \int_0^h k(x,s)\, f(s, u(s))ds, \quad x \in [0,h]$$

to have at least one nonnegative solution, then we apply this result to the boundary value problem.

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1. Introduction and Preliminaries

In this paper we consider the nonlinear Fredholm integral equation

$$u(x) = g(x) + \int_0^h k(x,s)\, f(s, u(s))ds, \quad x \in [0,h] \tag{1.1}$$

where $0 < h < \infty$. By solution of (1.1) we mean $u \in C[0,h]$ such that u satisfies (1.1). Our purpose is to present an existence result for (1.1) which guarantees the existence of at least one nonnegative solution $u \in [0,h]$. The theorem is proved using Krasnoselskii's Fixed Point Theorem, and it was implied us by the work of Maria Meehan and Donal O'Regan. Next, we shall apply this result to find conditions to existence of solution for the nonlinear boundary value problem. We close this paper with a numerical example.

Now, we remember the Krasnoselskii's Fixed Point Theorem

Theorem 1.1. *Let X be a Banach space, and let $K \subset X$ be a cone in X. Suppose $\Omega_1, \Omega_2 \in P_d(X)$ with $0 \in \Omega_1, \overline{\Omega}_1 \in \Omega_2$, and consider $T : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ a completely continuous operator such that either*

$$\|T(x)\| \leq \|x\| \text{ for } x \in K \cap \Omega_1 \text{ and } \|T(x)\| \geq \|x\| \text{ for } x \in K \cap \Omega_2$$

or

$$\|T(x)\| \geq \|x\| \text{ for } x \in K \cap \Omega_1 \text{ and } \|T(x)\| \leq \|x\| \text{ for } x \in K \cap \Omega_2$$

is true. Then T has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

2. Nonnegative solution for Fredholm integral equation

In the following, we consider the nonlinear integral equation (1.1).

The next result establishes what conditions are requires on g, k and f in order for (1.1) to have at least one nonnegative solution, i.e. $u \in C[0,h]$ which satisfies (1.1) and $u(x) \geq 0$ for any $x \in [0,h]$.

Theorem 2.1. *Let us consider the Fredholm integral equation (1.1) Assume that*

(i) $0 \leq k_x(s) = k(x,s) \in L^1[0,h]$ *for any* $x \in [0,h]$;

(ii) the map $x \mapsto k_x$ *is continuous from* $[0,h]$ *to* $L^1[0,h]$;

(iii) there are $M \in (0,1)$, $\widetilde{k} \in L^1[0,h]$, *and an interval* $[a,b] \subset [0,h]$ *such that* $k(x,s) \geq M\widetilde{k}(s) \geq 0$ *for any* $x \in [0,h]$ *and always everywhere in* $[0,h]$;

(iv) $k(x,s) \leq \widetilde{k}(s)$ *for any* $x \in [0,h]$ *and always everywhere* $s \in [0,h]$;

(v) $g \in C[0,h]$ *with* $g(x) \geq 0$ *for any* $x \in [0,h]$ *and* $\min\limits_{a \leq x \leq b} g(x) \geq M\|g\| = M \sup\limits_{0 \leq x \leq h} |g(x)|$;

(vi) $f : [0,h] \times R \to R$ *is continuous and there are the nondecreasing mappings* $\varphi, \psi \in C([0,h], R_+)$ *such that* $0 < \varphi(s) \leq f(x,s) \leq \psi(s)$, $x \in [0,h]$ *and always everywhere in* $[0,h]$;

(vii) there is $\alpha > 0$ *such that* $\dfrac{\alpha}{\|g\| + K_1\psi(\alpha)} > 1$, *where* $K_1 = \sup\limits_{0 \leq x \leq h} \int_0^h k(x,s)ds > 0$;

(viii) there exists $\beta > 0$, $\alpha \neq \beta$, *and* $x \in [0,h]$ *such that*

$$\frac{\beta}{g(x_0) + \varphi(M\beta)\int_a^b k(x_0,s)ds} < 1;$$

Then, equation (1.1) has at least one nonnegative solution, $u \in C[0,h]$. *Moreover, either*

$$0 < \alpha \leq \|u\| \leq \beta \text{ and } u(x) \geq M\alpha \text{ for } x \in [a,b] \text{ if } \alpha < \beta \tag{A}$$

or

$$0 < \beta \leq \|u\| \leq \alpha \text{ and } u(x) \geq M\beta \text{ for } x \in [a,b] \text{ if } \alpha > \beta \tag{B}$$

holds.

Proof: We define the operator $T : K \to K$ by $T(u)(x) = g(x) + \int_0^h k(x,s) f(s,u(s))ds$, $x \in [0,h]$ where $K = \left\{ u \in C[0,h];\, u(x) \geq 0,\ x \in [0,h];\ \min\limits_{a \leq x \leq b} u(x) \geq M\|u\| \right\}$.

The hypothesis (i)-(vi) imply that T is continuous and completely continuous.

Let $\Omega_1 = \{u \in C[0,h];\ \|x\| < \alpha\}$ and $\Omega_2 = \{u \in C[0,h];\ \|x\| < \beta\}$

Suppose that $u \in K \cap \partial\Omega_1$, i.e. $u(x) \geq 0,\ x \in [0,h]$ and $\|g\| = \alpha$. We have

$$\|T(u)\| \leq \|g\| + \psi(\|u\|) \sup_{0 \leq x \leq h} \int_0^h k(x,s)ds \leq \|g\| + K_1\psi(\alpha) \leq \alpha = \|u\|.$$

So, $\|T(u)\| < \|u\|$ for any $u \in K \cap \partial\Omega_1$.

Now, we consider $u \in K \cap \partial\Omega_2$, i.e. $u(x) \geq 0,\ x \in [0,h]$ and $\|u\| = \beta$. For any $x \in [a,b]$ it is satisfies the inequality $M\beta \leq u(x) \leq \beta$. We have

$$T(u)(x) \geq g(x_0) + \int_a^b k(x_0,s)\varphi(u(s))ds \geq g(x_0) + \varphi(M\beta)\int_a^b k(x_0,s)ds > \beta = \|u\|.$$

Hence $\|T(u)\| > \|u\|$ for any $u \in K \cap \partial\Omega_2$.

By Theorem 1.1., there is $u \in K$ a solution for the equation $T(u) = u$. This is equivalent with existence of a nonnegative solution for (1.1). In particular, if $\alpha < \beta$ than $u \in K \cap (\overline{\Omega}_2 \setminus \Omega_1)$, while $u \in K \cap (\overline{\Omega}_1 \setminus \Omega_2)$ if $\beta < \alpha$.

3. Nonnegative solution for boundary value problem

It's know that the boundary value problem

$$\begin{cases} -u'' = f(x,u), & x \in (0,h) \\ u(0) = u(h) = l \end{cases} \tag{3.1}$$

with $-\infty < l < \infty$, is equivalent to the integral equation

$$u(x) = l + \int_0^h G(x,s)\, f(s,u(s))ds,\ \ x \in [0,h] \tag{3.2}$$

For equation (3.2) we apply Theorem 2.1. and obtain the following result.

Theorem 3.1. *Suppose that:*

(ix) $f : [0,h] \times R \to R$ *is continuous and there are the nondecreasing mappings* $\varphi, \psi \in C([0,h]; R_+)$ *such that* $0 \leq \varphi(s) \leq f(x,s) \leq \psi(s)$ for any $x \in [0,h]$ and allways everywhere $s \in [0,h]$

(x) there is $\alpha > 0$ *such that* $\frac{8\alpha}{8l + h^2\psi(\alpha)} > 1$;

(xi) there exists $\beta > 0, \alpha \neq \beta$, *and* $x_0 \in [0,h]$ *such that* $\frac{2\beta}{2l + \varepsilon(h-\varepsilon)\varphi(M\beta)} < 1$;

Then, the problem (3.1) has at least one nonnegative solution $u \in C[0,h]$ *for which, either*

$$0 < \alpha \leq \|u\| \leq \beta \text{ and } u(x) \geq M\alpha \text{ for } x \in \left[\frac{h}{2} - \varepsilon, \frac{h}{2} + \varepsilon\right] \text{ if } \alpha < \beta \tag{A}$$

or

$$0 < \beta \le \|u\| \le \alpha \text{ and } u(x) \ge M\beta \text{ for } x \in \left[\frac{h}{2}-\varepsilon, \frac{h}{2}+\varepsilon\right] \text{ if } \alpha > \beta \tag{B}$$

holds.

Proof. The problem (3.1) is equivalent with the integral equation (3.2) where $G : [0,h] \times [0,h] \to R$, the Green function for (3.1), is given by

$$G(x,s) = \begin{cases} \dfrac{x(h-s)}{h}, & x \le s \le h \\ \dfrac{s(h-x)}{h}, & 0 \le s \le x \end{cases}$$

We shall apply Theorem 2.1. for equation (3.2). For this, we choose $g \equiv l \ge 0$; $k(x,s) = G(x,s)$ and $f := f$.

For $G(x,s)$ the hypothesis (i) and (ii) are satisfies. We have

$$0 \le M\frac{s(h-s)}{h} \le G(x,s) \le \frac{s(h-s)}{h},$$

for any $x \in \left[\frac{h}{2}-\varepsilon, \frac{h}{2}+\varepsilon\right]$ and $s \in [0,h]$, where $\varepsilon \in \left(0, \frac{h}{2}\right)$ and $0 < M \le \frac{1}{2} - \frac{\varepsilon}{h} < 1$.

Therefore, (iii) and (iv) from Theorem 2.1., are verify for $M \in \left(0, \frac{1}{2} - \frac{\varepsilon}{h}\right]$, $\bar{k} = \frac{s(h-s)}{h}$ for any $s \in [0,h]$ and $[a,b] = \left[\frac{h}{2}-\varepsilon, \frac{h}{2}+\varepsilon\right] \subset [0,h]$ with $\varepsilon \in \left(0, \frac{h}{2}\right)$. The condition (v) is obvious, and (vi) is the same with (ix).

We have $K_1 = \sup\limits_{0<x\le h} \left\{ \int\limits_0^x \frac{s(h-x)}{h} ds + \int\limits_x^h \frac{x(h-s)}{h} ds \right\} = \sup\limits_{0<x\le h} \frac{x(h-x)}{h} = \frac{h^2}{8}$. Than, (vii) is equivalent with existence to the real number $\alpha > 0$ such that $\frac{\alpha}{l + \frac{h^2}{8}\psi(\alpha)} > 1$. Hence, (x) is satisfies. Now, let $x_0 = \frac{h}{2}$. Than $\int\limits_{\frac{h}{2}-\varepsilon}^{\frac{h}{2}+\varepsilon} G\left(\frac{h}{2}, s\right) ds = \frac{\varepsilon(h-\varepsilon)}{2}$. Therefore, condition (viii) is satisfies if there is $\beta > 0, \alpha \neq \beta$ such that $\frac{2\beta}{2l+\varepsilon(h-\varepsilon)\omega(M\beta)} < 1$. Now, the conclusion is implies by Theorem 2.1.

In a same way, we can prove the next two results

Theorem 3.2. *Let as consider the nonlinear boundary value problem*

$$\begin{cases} -u'' = f(u), & x \in (0,h) \\ u(0) = u(h) = 0 \end{cases} \tag{3.3}$$

Suppose that:

(xii) $f : R \to R$ is continuous and nondecreasing with $f(x) > 0$ for $x > 0$;

(xiii) exists $\varepsilon \in \left(0, \frac{h}{2}\right)$, $M \in (0,1)$ and with $\alpha \neq \beta$ such that

$$\frac{\beta}{f(M\beta)} < \frac{\varepsilon(h-\varepsilon)}{2} \leq \frac{h^2}{8} < \frac{\alpha}{f(\alpha)} \tag{3.4}$$

Then, the problem (3.3) has at least one nonnegative solution $u \in C[0,h]$ for which, either

$$0 < \alpha \leq \|u\| \leq \beta \text{ and } u(x) \geq M\alpha \text{ for } x \in \left[\frac{h}{2} - \varepsilon, \frac{h}{2} + \varepsilon\right] \text{ if } \alpha < \beta \tag{A}$$

or

$$0 < \beta \leq \|u\| \leq \alpha \text{ and } u(x) \geq M\beta \text{ for } x \in \left[\frac{h}{2} - \varepsilon, \frac{h}{2} + \varepsilon\right] \text{ if } \alpha > \beta \tag{B}$$

holds

Theorem 3.3. *Assume that:*

(xiv) $f : R \to R$ is continuous and nondecreasing with $f(x) > 0$ for $x > 0$;;

(xv) exists $\alpha, \beta > 0$ with $\alpha \neq \beta$ such that

$$\frac{\beta}{f\left(\frac{1}{2}\beta\right)} < \frac{1}{2} \cdot \frac{h^2}{8} \leq \frac{h^2}{8} < \frac{\alpha}{f(\alpha)} \tag{3.5}$$

Then, the problem (3.3) has at least one nonnegative solution, $u \in C[0,h]$. Moreover, either

$$0 < \alpha \leq \|u\| \leq \beta \text{ and } u(x) \geq \frac{1}{2}\alpha \text{ for } x \in [a,b] \subset \left(0, \frac{h}{2}\right) \text{ if } \alpha < \beta \tag{A}$$

or

$$0 < \beta \leq \|u\| \leq \alpha \text{ and } u(x) \geq \frac{1}{2}\beta \text{ for } x \in [a,b] \subset \left(0, \frac{h}{2}\right) \text{ if } \alpha > \beta \tag{B}$$

holds

4. Numerical example

In the following, we apply the results of this section to the nonlinear boundary value problem

$$\begin{aligned} u'' + \frac{1}{4}u^2 + u = 0, \\ u(0) = u(1) = 1 \end{aligned} \tag{4.1}$$

The mapping $f : R \to R$ given by $f(x) = \frac{1}{4}x^2 + x$ is continuous and nondecreasing with $f(x) > 0$ for $x > 0$. Inequality (x) became $\frac{8\alpha}{8 + f(\alpha)} > 1$ with solution

$\alpha \in (14 - 2\sqrt{41}.14 + 2\sqrt{41})$. Inequality (xi) became $\frac{\beta}{1 + \frac{1}{16} f\left(\frac{1}{2}\beta\right)} < 1$. This inequality has the solution $\beta \in (0.124 - 12\sqrt{105}) \cup (124 + 12\sqrt{105}, \infty)$.

Our purpose is to approximate the solution $u \in C[0,1]$ of (4.1) using the iteration method. For this, it is important the selection of the first function $u_0 \in C^2[0,1]$. By results of this section, we know that $\alpha \leq u(x) \leq \beta$ or $\beta \leq u(x) \leq \alpha$ every where in $[0,1]$. So, we start with $u_0(x) = 1 + \frac{4}{5}x(1-x)$, $x \in [0,1]$ for which we have $\|u_0\| = 1.2 > 14 - 2\sqrt{41}$. By the first iteration, obtain

$$u_1(x) = 1 + \int_0^1 G(x,s) f(u_0(s)) ds = 1 - \frac{2}{375}x^6 + \frac{2}{125}x^5 + \frac{13}{150}x^4 - \frac{1}{5}x^3 - \frac{5}{8}x^2 + \frac{2183}{3000}x$$

Next, we approximate u_1 with $\overline{u}_1(x) = 1 + 0.7536x - 0.7536x^2$. From a new iteration, results

$$u_2(x) = 1 - 0.00473x^6 + 0.01419x^5 + 0.08236x^4 - 0.1884x^3 - 0.625x^2 + 0.72157x.$$

For u_2 we have $\max\limits_{0 \leq x \leq 1} \left| u_2''(x) + \frac{1}{4}u_2^2(x) + u_2(x) \right| = 0.004$ so, u_2 is a good approximation for a solution to (4.1).

REFERENCES

[1] M. KRASNOSELSKII, Positive Solution of Operator Equation, Noordhoff, 1964

[2] M. MEEHAN & D. O'REGAN, Multiple Nonnegative Solutions of Nonlinear Integral Equation on Compact and Semi-Infinite Intervals, Applicable Analysis, 74, 413-427

[3] R. PRECUP, Nonlinear Integral Equations (Romanian), Babeş-Bolyai Univ, Cluj-Napoca, 1993.

[4] R. PRECUP, On Some Fixed Point of Deimling, Nonlinear Analysis, Theory & Application, 23(1994), 1315-1320.

[5] I.A. RUS, Differentials equations, integrals equations and dynamic systems (Romanian), Transilvania Press, Cluj-Napoca, 1996

[6] E. ZEIDLER Nonlinear Functional Analysis and its Applications, Springer-Verlag, New York - Berlin - Heidelberg - Tokyo, 1985.

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