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A REGULARIZATION METHOD FOR THE NUMERICAL SOLUTION OF THE CAUCHY PROBLEM FOR THE HELMHOLTZ EQUATION

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Abstract. In this paper, the iterative algorithm proposed by Kozlov et al. [12] for obtaining approximate solutions to the ill-posed Cauchy problem for the Helmholtz equation is analysed. The technique is then numerically implemented using the boundary element method (BEM). The numerical results confirm that the iterative BEM produces a convergent and stable numerical solution with respect to increasing the number of boundary elements and decreasing the amount of noise added into the input data. An efficient stopping regularising criterion is also proposed.

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Keywords: Cauchy problem, Helmholtz equation, boundary element method

1. Introduction

The Helmholtz equation arises naturally in many physical applications related to wave propagation and vibration phenomena. It is often used to describe the vibration of a structure [1], the acoustic cavity problem [2], the radiation wave [3] and the scattering of a wave [4]. Another important application of the Helmholtz equation is the problem of heat conduction in fins, see e.g. Kern and Kraus [5], and we focus on the latter problem in this study. The knowledge of the Dirichlet, Neumann or mixed boundary conditions on the entire boundary of the solution domain gives rise to direct problems for the Helmholtz equation whose well-posedness via the removal of the eigenvalues of the Laplacian operator are well established, see e.g. Chen and Zhou [6]. Unfortunately, many engineering problems do not belong to this category. In particular, the boundary conditions are often incomplete, either in the form of underspecified and overspecified boundary conditions on different parts of the boundary or the solution is prescribed at some internal points in the domain. These are inverse problems, and it is well known that they are generally ill-posed, i.e. the existence, uniqueness and stability of their solutions are not always guaranteed. There are important studies in the literature of the Cauchy problem for the Helmholtz equation. Unlike in direct problems, the uniqueness of the Cauchy problem is guaranteed without the necessity of removing the eigenvalues for the Laplacian. However, the Cauchy problem suffers from the non-existence and instability of the solution. A BEM-based acoustic holography technique using the singular value decomposition (SVD) for the reconstruction of sound fields generated by irregularly shaped sources has been developed by Bai [7]. The vibrational velocity, sound pressure and acoustic power on the vibrating boundary comprising an enclosed space have been reconstructed by Kim and Ih [8] who have used the SVD in order to obtain the inverse solution in the least-squares sense and to express the acoustic modal expansion between the measurement and source field. Wang and Wu [9] have developed a method employing the spherical wave expansion theory and a least-squares minimisation to reconstruct the acoustic pressure field from a vibrating object and their method has been extended to the reconstruction of acoustic pressure fields inside the cavity of a vibrating object by Wu and Yu [10]. Recently, DeLillo et al. [11] have detected the source of acoustical noise inside the cabin of a midsize aircraft from measurements of the acoustical pressure field inside the cabin by solving a linear Fredholm integral equation of the first kind. At this stage, it is worth mentioning that in the above papers the application of traditional methods, such as the Tikhonov regularisation or the truncated SVD, can solve Cauchy problems which may not have a solution and this seems redundant. Therefore, we have decided in this study to use an iterative BEM algorithm for the solution of a Cauchy problem of a Helmholtz-type equation based on an alternating iterative procedure which consists of obtaining successive solutions to well-posed mixed boundary value problems, similar to that proposed by Kozlov et al. [12]. The strength of this iterative algorithm is that it is convergent if and only if the solution of the Cauchy problem exists, which overcomes the previous mathematical redundancy. Whilst Kozlov et al. [12] proved the mathematical convergence of the algorithm without actually finding the solution, the aim of this paper is to show the numerical stability and convergence of the present algorithm. Furthermore, in order to cease the iterative procedure before the effects of the accumulation of noise become dominant, and the errors in the numerical solution start increasing, a stopping criterion is also proposed.

2. Mathematical Formulation

Referring to heat transfer for the sake of the physical explanation, we assume that the temperature field $T(\underline{x})$ satisfies the Helmholtz equation in an open bounded domain $\Omega \subset R^d$, where d is the dimension of the space in which the problem is posed, usually $d \in \{1, 2, 3\}$, namely

$$\mathcal{L}T(\underline{x}) \equiv (\Delta + k^2)T(\underline{x}) = 0, \qquad \underline{x} \in \Omega, \tag{1}$$

where $k = \alpha + i\beta \in C$, $i = \sqrt{-1}$. For example, when $\alpha = 0.0$ and $\beta \in R$, the partial differential equation (1) models the heat conduction in a fin, see e.g. Kern and Kraus [5], Manzoor *et al.* [6] and Wood *et al.* [7], where T is the dimensionless local fin temperature, $\beta^2 = h/(kt)$, h is the surface heat transfer coefficient $[\mathrm{W}/(\mathrm{m}^2\ \mathrm{K})]$, k is the thermal conductivity of the fin $[\mathrm{W}/(\mathrm{m\ K})]$ and t is the half-fin thickness [m]. We now let $n(\underline{x})$ be the outward normal vector at the boundary $\Gamma = \partial\Omega$ and $\Phi(\underline{x})$ be the flux at a point $\underline{x} \in \Gamma$ defined by

$$\Phi(\underline{x}) = \frac{\partial T}{\partial n}(\underline{x}), \qquad \underline{x} \in \Gamma. \tag{2}$$

The Cauchy problem under investigation requires solving the partial differential equation (1) subject to the boundary conditions

$$T(\underline{x}) = \widetilde{T}(\underline{x}), \qquad \Phi(\underline{x}) = \widetilde{\Phi}(\underline{x}), \qquad \underline{x} \in \Gamma_2, \tag{3}$$

where $\widetilde{T}$ and $\widetilde{\Phi}$ are prescribed functions and $\Gamma_2 \subset \Gamma$, meas$(\Gamma_2) > 0$. In the above formulation of the boundary conditions (3), it can be seen that the boundary Γ_2 is overspecified by prescribing both the temperature $T|_{\Gamma_2}$ and the flux $\Phi|_{\Gamma_2}$, whilst the boundary $\Gamma_1 = \Gamma \setminus \Gamma_2$ is underspecified since both the temperature $T|_{\Gamma_1}$ and the flux $\Phi|_{\Gamma_1}$ are unknown and have to be determined.

3. Description of the Algorithm

Step 1. Specify an initial approximation $\Phi^{(0)}(\underline{x})$ for the flux on Γ_1 and solve the well-posed mixed boundary value problem

$$\begin{cases} LT^{(1)}(\underline{x}) = 0, & \underline{x} \in \Omega \\ \Phi^{(1)}(\underline{x}) \equiv \dfrac{\partial T^{(1)}}{\partial n}(\underline{x}) = \Phi^{(0)}(\underline{x}), & \underline{x} \in \Gamma_1 \\ T^{(1)}(\underline{x}) = \tilde{T}(\underline{x}), & \underline{x} \in \Gamma_2 \end{cases} \tag{4}$$

in order to determine $T^{(1)}(\underline{x})$ for $\underline{x} \in \Omega$ and $T^{(1)}(\underline{x})$ for $\underline{x} \in \Gamma_1$.

Step 2. Having constructed the approximation $T^{(2n-1)}(\underline{x})$, $n \geq 1$, the well-posed mixed boundary value problem

$$\begin{cases} LT^{(2n)}(\underline{x}) = 0, & \underline{x} \in \Omega \\ T^{(2n)}(\underline{x}) = T^{(2n-1)}(\underline{x}), & \underline{x} \in \Gamma_1 \\ \Phi^{(2n)}(\underline{x}) \equiv \dfrac{\partial T^{(2n)}}{\partial n}(\underline{x}) = \tilde{\Phi}(\underline{x}), & \underline{x} \in \Gamma_2 \end{cases} \tag{5}$$

is solved to determine $T^{(2n)}(\underline{x})$ for $\underline{x} \in \Omega$ and $\Phi^{(2n)}(\underline{x}) \equiv \dfrac{\partial T^{(2n)}}{\partial n}(\underline{x})$ for $\underline{x} \in \Gamma_1$.

Step 3. Having constructed the function $T^{(2n)}(\underline{x})$, $n \geq 1$, the well-posed mixed boundary value problem

$$\begin{cases} LT^{(2n+1)}(\underline{x}) = 0, & \underline{x} \in \Omega \\ \Phi^{(2n+1)}(\underline{x}) \equiv \dfrac{\partial T^{(2n+1)}}{\partial n}(\underline{x}) = \Phi^{(2n)}(\underline{x}), & \underline{x} \in \Gamma_1 \\ T^{(2n+1)}(\underline{x}) = \tilde{T}(\underline{x}), & \underline{x} \in \Gamma_2 \end{cases} \tag{6}$$

is solved to determine $T^{(2n+1)}(\underline{x})$ for $\underline{x} \in \Omega$ and $T^{(2n+1)}(\underline{x})$ for $\underline{x} \in \Gamma_1$.

Step 4. Repeat steps 2 and 3 until a prescribed stopping criterion is satisfied.

<u>**Remarks:**</u> Let $H^1(\Omega)$ be the Sobolev space and $H^{1/2}(\Gamma)$ be the space of traces on Γ corresponding to $H^1(\Omega)$, see e.g. Lions and Magenes [13]. We denote by $H^{1/2}(\Gamma_j)$ the space of functions from $H^{1/2}(\Gamma)$ that are bounded on Γ_j and by $H^{1/2}(\Gamma_j)^*$ the dual space of $H^{1/2}(\Gamma_j)$, for $j = 1, 2$. Kozlov *et al.* [12] showed that if Γ is smooth, $\tilde{T} \in H^{1/2}(\Gamma_2)$, $\tilde{\Phi} \in H^{1/2}(\Gamma_2)^*$ and k is purely imaginary, i.e. $\alpha = 0$, then the alternating algorithm based on steps 1 – 4 produces two sequences of approximate solutions $\{T^{(2n)}(\underline{x})\}_{n\geq 1}$ and $\{T^{(2n-1)}(\underline{x})\}_{n\geq 1}$ which both converge in $H^1(\Omega)$ to the solution $T(\underline{x})$ of the Cauchy problem (1) and (3), if it exists, for any initial guess $\Phi^{(0)} \in H^{1/2}(\Gamma_1)^*$. Also the same conclusion is obtained if at step 1 we specify an initial guess $T^{(0)} \in H^{1/2}(\Gamma_1)$, instead of

an initial guess for the flux $\Phi^{(0)} \in H^{1/2}(\Gamma_1)^*$, and we modify accordingly the steps $1-3$ of the algorithm. We note that if the initial guess $\Phi^{(0)}$ is in $H^{1/2}(\Gamma_1)^*$ and the boundary data $\widetilde{T}$ and $\widetilde{\Phi}$ are in $H^{1/2}(\Gamma_2)$ and $H^{1/2}(\Gamma_2)^*$, respectively, the problems (4)–(6) are well-posed and solvable in $H^1(\Omega)$, provided that k^2 is not an eigenvalue of the Laplacian operator Δ, see Lions and Magenes [13]. These intermediate mixed well-posed problems (4)–(6) are solved using the BEM described in the next section.

4. Boundary Element Method

The Helmholtz equation (1) can also be formulated in integral form, see e.g. Chen and Zhou [6], as

$$c(\underline{x})T(\underline{x}) + \int_\Gamma \frac{\partial E(\underline{x},\underline{y})}{\partial n(\underline{y})} T(\underline{y})\, d\Gamma(\underline{y}) = \int_\Gamma E(\underline{x},\underline{y})\Phi(\underline{y})\, d\Gamma(\underline{y}) \tag{7}$$

for $\underline{x} \in \overline{\Omega} = \Omega \cup \Gamma$, where the first integral is taken in the sense of the Cauchy principal value, $c(\underline{x}) = 1$ for $\underline{x} \in \Omega$ and $c(\underline{x}) = 1/2$ for $\underline{x} \in \Gamma$ (smooth), and E is the fundamental solution for the Helmholtz equation (1), which in two-dimensions is given by

$$E(\underline{x},\underline{y}) = \frac{i}{4} H_0^{(1)}\Big(k\, r(\underline{x},\underline{y})\Big). \tag{8}$$

Here $r(\underline{x},\underline{y})$ represents the distance between the load point $\underline{x}$ and the field point $\underline{y}$ and $H_0^{(1)}$ is the Hankel function of order zero of the first kind. It should be noted that in practice the boundary integral equation (7) can rarely be solved analytically and thus a numerical approximation is required.

A BEM with constant boundary elements is used in order to solve the intermediate mixed well-posed boundary value problems resulting from the iterative method adopted, which is described in Section 3. Consequently, the boundary Γ is approximated by N straight line segments in a counterclockwise sense along with the temperature and the flux which are considered to be constant and take their values at the midpoint, i.e. the collocation point, also known as the node, of each element. More specifically, we have

$$\begin{aligned} &\Gamma \approx \bigcup_{n=1}^{N} \Gamma_n, \quad \Gamma_n = [\underline{y}^{n-1}, \underline{y}^n], \quad n = 1, \cdots, N \\ &\underline{y}^N = \underline{y}^0, \qquad \underline{x}^n = (\underline{y}^{n-1} + \underline{y}^n)/2, \quad n = 1, \cdots, N \end{aligned} \tag{9}$$

and

$$T(\underline{y}) = T(\underline{x}^n), \quad \Phi(\underline{y}) = \Phi(\underline{x}^n), \quad \underline{y} \in \Gamma_n, \quad n = 1, \cdots, N. \tag{10}$$

By applying the boundary integral equation (7) at each collocation point $\underline{x}^m$, $m = 1, \cdots, N$, and taking into account the fact that the boundary is always smooth at these points, we arrive at the following system of linear algebraic equations

$$AT = B\Phi \tag{11}$$

where A and B are matrices which depend solely on the geometry of the boundary Γ and the vectors $\underline{T}$ and $\underline{\Phi}$ consist of the discretised values of the temperature and the flux on the boundary Γ, namely

$$T(m) = T(\underline{x}^m), \qquad \Phi(m) = \Phi(\underline{x}^m) \tag{12}$$

for $m = 1, \cdots, N$, and

$$A(n,m) = \delta_{nm}/2 + \int_{\Gamma_n} \frac{\partial E(\underline{x}^m, y)}{\partial n(y)} \, d\Gamma(y), \quad B(n,m) = \int_{\Gamma_n} E(\underline{x}^m, \underline{y}) \, d\Gamma(\underline{y}) \tag{13}$$

for $\underline{x}^m \in \widetilde{\Gamma}$ and m, $n = 1, \cdots, N$, where δ_{nm} is the Kronecker tensor. We note that the sense of the Cauchy principal value assigned to the first integral in the boundary integral equation (7) has meaning only when $\underline{x}^m \in \Gamma_n$, as in the other cases the integral is non-singular. If the boundaries Γ_1 and Γ_2 are discretised into N_1 and N_2 boundary elements, respectively, such that $N_1 + N_2 = N$, then equation (11) represents a system of N linear algebraic equations with $2N$ unknowns. The discretisation of the boundary conditions (3) provides the values of $2N_2$ of the unknowns and the problem reduces to solving a system of N equations with $2N_1$ unknowns which can be generically written as

$$C\underline{X} = \underline{F} \tag{14}$$

where $\underline{F}$ is computed using the boundary conditions (3), the matrix C depends solely on the geometry of the boundary Γ and the vector $\underline{X}$ contains the unknown values of the temperature and the flux on the boundary Γ_1.

5. Numerical Results and Discussion

In this section we illustrate the numerical results obtained using the alternating boundary element algorithm proposed in Section 3 combined with the BEM described in Section 4.

5.1. Example

In order to present the performance of the numerical method proposed, we solve the Cauchy problem for a typical benchmark example in a two-dimensional smooth geometry, namely, the annular domain $\Omega = \{\underline{x} = (x_1, x_2) \mid R_i^2 < x_1^2 + x_2^2 < R_o^2\}$, $R_i = 0.5$ and $R_o = 1.0$. We assume that the boundary Γ of the solution domain is divided into two disjointed parts, namely $\Gamma_1 = \{\, \underline{x} \in \Gamma \mid x_1^2 + x_2^2 = R_i^2\}$ and $\Gamma_2 = \{\underline{x} \in \Gamma \mid x_1^2 + x_2^2 = R_o^2\}$. We consider the following analytical solution for the temperature:

$$T^{(an)}(\underline{x}) = \exp(a_1 x_1 + a_2 x_2), \qquad \underline{x} = (x_1, x_2) \in \overline{\Omega}, \tag{15}$$

where $k = \alpha + i\beta$, $\alpha = 0$, $\beta = 2.0$, $a_1 = 1.0$ and $a_2 = -\sqrt{\beta^2 - a_1^2}$.

This example has a flux on the boundary Γ given by

$$\Phi^{(an)}(\underline{x}) = [a_1 n_1(\underline{x}) + a_2 n_2(\underline{x})] T^{(an)}(\underline{x}), \quad \underline{x} = (x_1, x_2) \in \Gamma. \tag{16}$$

The Cauchy problems given by equations (1) and (3) has been solved iteratively using the BEM to provide simultaneously the unspecified boundary temperature and flux on Γ_1 and the temperature inside the solution domain. The number of constant boundary elements used for discretising the boundary Γ was taken to be $N_1 = N_2 = N/2$.

5.2. Initial Guess

An arbitrary function $\Phi^{(0)} \in H^{1/2}(\Gamma_1)^*$ may be specified as an initial guess for the flux on Γ_1, and we have taken

$$\Phi^{(0)}(\underline{x}) = 0, \qquad \underline{x} \in \Gamma_1. \tag{17}$$

5.3. Convergence of the Algorithm

In order to investigate the convergence of the algorithm, at every iteration we evaluate the accuracy errors defined by

$$e_T = \|T^{(n)} - T^{(an)}\|_{L^2(\Gamma_1)}, \qquad e_\Phi = \|\Phi^{(n)} - \Phi^{(an)}\|_{L^2(\Gamma_1)}, \tag{18}$$

where $T^{(n)}$ and $\Phi^{(n)}$ are the temperature and the flux on the boundary Γ_1 retrieved after n iterations, respectively, and each iteration consists of solving the two mixed well-posed problems (5) and (6).

For the inverse problem we use "exact boundary data", i.e. boundary data obtained by solving a direct well-posed problem, namely

$$\begin{cases} LT(\underline{x}) = 0, & \underline{x} \in \Omega \\ \Phi(\underline{x}) \equiv \dfrac{\partial T}{\partial n}(\underline{x}) = \Phi^{(an)}(\underline{x}), & \underline{x} \in \Gamma_1 \\ T(\underline{x}) = T^{(an)}(\underline{x}), & \underline{x} \in \Gamma_2. \end{cases} \tag{19}$$

5.4. Stopping Criterion

Once the convergence with respect to increasing N of the numerical solution to the exact solution has been established, we fix $N = 40$ and investigate the stability of the numerical solution by adding Gaussian random noise $p \in \{0, 1, 2\}\%$ into the temperature data $\tilde{T}|_{\Gamma_2}$. As p decreases then e_T and e_Φ decrease. However, the errors in predicting the temperature and the flux on the underspecified boundary Γ_1 decrease up to a certain iteration number and after that they start increasing. If the iterative process is continued beyond this point then the numerical solutions lose their smoothness and become highly oscillatory and unbounded, i.e. unstable. Therefore, a regularising stopping criterion must be used in order to terminate the iterative process at the point where the errors in the numerical solutions start increasing. If we evaluate the Euclidean norm of the vector $C\underline{X} - \underline{F}$ then this should tend to zero as $\underline{X}$ tends to the exact solution. Hence after each iteration we evaluate the error

$$E = \|C\underline{X}^{(n)} - \underline{F}\|_2, \tag{20}$$

where $\underline{X}^{(n)}$ is the vector obtained from the values of the temperature and the flux on the boundary Γ_1 retrieved after n iterations. The error E includes information on both the temperature and the flux and it is expected to provide an appropriate stopping criterion.

A natural stopping criterion is given by the discrepancy principle which ceases the iterative procedure described above for

$$n \in N : \qquad E = \|CX^{(n)} - F\|_2 \le \varepsilon, \tag{21}$$

where ε is a measure of the level of noise in the measurements of the data on the overdetermined boundary Γ_2, see Morozov[14].

5.5. Stability of the Algorithm

Based on the stopping criterion described in Subsection 5.4, Table 1 presents the errors e_T and e_Φ and the value of ε given by the discrepancy principle obtained with $N = 40$ boundary elements and various amounts of noise added into the input data $T|_{\Gamma_1}$, namely $p \in \{0, 1, 2\}$. From this table it can be seen that the numerical solution converges to the exact solution as the level of noise, p, added into the input boundary data decreases. From the numerical results presented in this section, it can be concluded that the stopping criterion developed in Subsection 5.4 has a regularising effect and the numerical solution obtained by the iterative BEM described in this paper is convergent and stable with respect to increasing the mesh size discretisation and decreasing the level of noise added into the input data, respectively.

Although not illustrated here, an important conclusion is reported, namely, that the alternating iterative algorithm described in Section 3 is not convergent for the differential operator $L \equiv \Delta + k^2$ for $k = \alpha + i\beta$, $\alpha \in R$ and $\beta = 0$, both errors e_T and e_Φ defined by relations (18) "blow up" after the first iteration. The reason is that the proof of convergence of the iterative algorithm of Kozlov *et al.* [12] requires, as a necessary condition, $L \equiv \Delta + k^2$ to be positive-definite differential operators and this is not the case when k is real.

Table 1: The errors e_T and e_Φ and the value of ε.

Error	p=0%	p=1%	p=2%
e_T	4.02×10^{-4}	1.35×10^{-1}	2.27×10^{-1}
e_Φ	3.99×10^{-2}	5.71×10^{-1}	8.70×10^{-1}
ε	0.00	1.99×10^{-1}	3.98×10^{-1}

6. Conclusions

In this paper we have investigated the Cauchy problem for the Helmholtz equation in the two-dimensional case. In order to deal with the instabilities of the solution of this ill-posed problem, an iterative BEM was employed which reduced the Cauchy problem to solving a sequence of well-posed boundary value problems. A stopping criterion, necessary for ceasing the iterations at the point where the accumulation of noise becomes dominant and the errors in predicting the exact solution increase, has also been presented. The numerical results obtained for various numbers of boundary elements and various amounts of noise added to the input data showed that the BEM produces a convergent, stable and consistent numerical solution with respect to increasing the number of boundary elements and decreasing the amount of noise.

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