Bul. Ştiinţ. Univ. Baia Mare, Ser. B,
Matematică-Informatică, Vol. XVIII(2002), Nr. 2, 253 - 256

INTEGRAL OPERATORS PRESERVING SUBORDINATION AND HARDY CLASSES

Gheorghe MICLĂUŞ

Abstract: In this paper we obtain some results for Hardy classes of some integral operators preserving subordination.

MSC: 30C35, 30C45

Keywords: Hardy classes, integral operators preserving subordination

1. Introduction

Let be U unit disk, $H(U)$ space of analytic functions in U, $E \subset H(U)$ and $I : E \to H(U)$ integral operator. We say that I is preserving subordination if:

$$f \prec g \Rightarrow I[f] \prec I[g]. \tag{1}$$

In 1935, G.M. Goluzin considered the operator of Alexander $I : H_0 \to H(U)$ defined by:

$$I[f](z) = \int_0^z f(t)\, t^{-1}\, dt \tag{2}$$

when $H_0 = \{f \in H(U) \,|\, f(0) = 0\}$. He showed that if g is in convex than (1) is satisfied.

In 1970, T. Suffrige extended this result to the case when g is starlike.

In 1981, S.S. Miller and P.T. Mocanu [4] using differential subordination theory, extended this results considering the operator $I : H_0 \to H(U)$ defined by:

$$I[f](z) = \left[\int_0^z f^{\beta}(t)\, t^{-1}\, dt\right]^{\frac{1}{\beta}}. \tag{3}$$

He showed that if $\beta \geq 1$ and g is a starlike function this operator is preserving subordination.

In 1975, D. Halkenbeck and S. Rusheweyh showed that if $\gamma \neq 0$, $Re\gamma \geq 0$ and g is convex then the operator $I : H(U) \to H(U)$, defined by:

$$I[f](z) = \frac{1}{z^{\gamma}} \int_{0}^{z} f(t)\, t^{\gamma - 1}\, dt, \tag{4}$$

is preserving subordination.

In 1989, S.S. Miller and P.T. Mocanu considered the operator of Singh: $I : E \to H(U)$

$$I[f](z) = \left(\frac{1}{z^{\gamma}} \int_{0}^{z} f^{\beta}(t)\, t^{\gamma - 1}\, dt \right)^{\frac{1}{\beta}} \tag{5}$$

determined the conditions for preserving subordination.

2. Preliminaries

For $f \in H(U)$ and $z = re^{i\theta}$ we denote $M(r, f) = \left(\frac{1}{2\pi} \int_{0}^{2\pi} \left| f(re^{i\theta}) \right|^{p} d\theta \right)^{\frac{1}{p}}$, for $0 < p < \infty$

$$M_{\infty}(r, f) = \sup_{0 \leq \theta \leq 2\pi} \left| f(re^{i\theta}) \right|, for\, p = \infty.$$

A function is said to be of Hardy class H^{p}, $0 < p \leq \infty$ if $M(r, f)$ remains bounded as $r \to 1^{-}$. H^{∞} is the class of bounded analytic functions in the unit disk.

Let β and γ be complex constants with $Re\beta > 0$, $Re\gamma \geq 0$ and let $E_{\beta,\gamma}$ be defined as follows:

$$E_{\beta,\gamma} = \begin{cases} H(U), & if\ \beta = 1,\ \gamma \neq 0 \\ H_{0}\ , & if\ \beta = 1,\ \gamma = 0 \\ \left\{ f \in H(U) \mid f(z) = z^{j} h(z),\ h(z) \neq 0,\ j \geq 1 \right\} & if\ \frac{1}{\beta} \in \mathbb{N} - \{1\} \\ \left\{ f \in H_{0} \,\middle|\, f'(0) \neq 0,\ Re\left[\frac{\beta z f'(z)}{f(z)} + \gamma \right] > 0 \right\} & otherwise. \end{cases}$$

Note that $E_{\beta,0}$ with $\beta > 0$ and $\frac{1}{\beta} \notin \mathbb{N}$ is the class of starlike functions.

Lemma 1. [3]. Let $f \in E_{\beta,0}$ with $\beta > 0$ and let $g(z) = b_{1}z + b_{2}z^{2} + \ldots$ be starlike functions in U. If the operator $F : E_{\beta,0} \to H(U)$ is defined by:

$$F[f](z) = \left[\int_{0}^{z} f^{\beta}(t)\, t^{-1}\, dt \right]^{\frac{1}{\beta}}$$

then F is univalent and $f \prec g \Rightarrow F[f] \prec F[g]$.

Lemma 2. [3]. Let $Re\beta > 0$, $Re\gamma \geq 0$ and $f, g \in E_{\beta,\gamma}$ with $g'(0) \neq 0$. Let:

$$Re\left[(\beta - 1)\frac{zg'(z)}{g(z)} + \left(1 + \frac{zg''(z)}{g'(z)}\right)\right] > -\delta \tag{6}$$

when

$$\delta = \min\left\{Re\gamma, \frac{2Re\beta\, Re\gamma}{(|\beta + \gamma| + |\beta - \gamma|)^2}\right\}$$

If $I : E_{\beta,\gamma} \to H(U)$ is the operator (5) then $I[g]$ is univalent and $f \prec g \Rightarrow I[f] \prec I[g]$.

3. Main Results

Theorem 1. *If $f \in H_0$, g is a starlike function, $f \prec g$ and I is the Alexander's operator (2), then*

$$\text{(i) } I[f] \in H^{\frac{\lambda}{1-\lambda}},\ \ for\ all\ \lambda < \frac{1}{2};\ \text{(ii) } I^n(f) \in H^{\infty},\ n \geq 2\,(I^n = \underbrace{I \circ \ldots \circ I}_{n})$$

Proof. Since the Alexander operator preserves subordination we have $I[f] \prec I[g]$. Because g is starlike function from [1] we have $I[g] \in H^{\lambda}$, $\lambda < \frac{1}{2}$. Applying the subordination's theorem of Littlewood we obtain $I[f] \in H^{\lambda}$, $\lambda < \frac{1}{2}$. From theorem 7 [1] we have for the Alexander's operator $I^n(f) \in H^{\infty}$.

Theorem 2. *If $f \in H_0$, g is convex function and I is the Bernardi's operator: $I[f](z) = \frac{1+\gamma}{z^{\gamma}}\int\limits_0^z f(t)\,t^{\gamma-1}\,dt$, with $Re\gamma > 0$ and $f \prec g$, then $I^n(f) \in H^{\infty}$.*

Proof. Since the Bernardi's operator differs only by a constant from the operator (4), it has the same Hardy class as (4). Since g is convex function and from Theorem 7 [1] we have $I[g] \in H^{\infty}, hence I[f] \in H^{\infty}$.

If we consider the all analytical functions such that for any function from this class there exists a convex function to which it is subordinated, then the Bernadi's operator transform this class into the class of bounded functions.

Theorem 3. *Let $f \in H_0$ and $\beta \geq 1$, $\gamma = 0$ in the operator of Singh (5) and let g starlike function that $f \prec g$ then*

$$\text{(i) } if\ \beta > \frac{n}{2}\ then\ I^n[f] \in H^{\frac{\beta p}{\beta - np}},\ (\forall\, p)\ p < \frac{1}{2};\ \text{(ii) } if\ \beta \leq \frac{n}{2}\ then\ I^n[f] \in H^{\infty}$$

Proof. The operator of Singh for $\gamma = 0$ differs from the operator $F[f](z) = \left[\int\limits_0^z f^{\beta}(t)\,t^{-1}\,dt\right]$ by a constant so that they have the same Hardy class. If $\beta \geq I$ and g is starlike then F preserves subordination. From $f \prec g$ and $I(g) \in H^{\infty}$ $\lambda = \frac{\beta p}{\beta - p}$, $p < \frac{1}{2}$ we have

$I(f) \in H^{\lambda}$. From Theorem 7 [1] we obtain for $\beta > \frac{n}{2}$, $I^n[f] \in H^{\frac{\beta\lambda}{\beta - n\lambda}}$ and for $\beta \leq \frac{n}{2}$, $I^n[f] \in H^{\infty}$.

Theorem 4. *Let $f \in E_{\beta,0}$ with $\beta > 0$ and let $f \prec g$, $g(z) = b_1 z + b_2 z^2 + \ldots$ is starlike in U. If I is the operator of Singh then*

(i) *if* $\beta > \frac{n}{2}$ *then* $I^{n+1}(f) \in H^{\frac{\beta\lambda}{\beta - n\lambda}}$, $p < \frac{1}{2}$; (ii) *if* $\beta \leq \frac{n}{2}$ *then* $I^{n+1}(f) \in H^{\infty}$.

Proof. From Lemma 1 we have that for the operator $F[f](z) = \left[\int_0^z f^{\beta}(t)\, t^{-1}\, dt\right]^{\frac{1}{\beta}}$, $f \prec g \Rightarrow F(f) \prec G(g)$. Hence $F(f) \in H^{\lambda}$, $\lambda < \frac{1}{2}$. For the operator $B[f](z) = \left(\int_0^z f^{\beta}(t)\, t^{-1}\, dt\right)^{\frac{1}{\beta}}$ we have $B(f) \in H^{\lambda}$, $\lambda < \frac{1}{2}$. Let $I = A \circ B$ be the operator of Singh where $A[f](z) = \left(\frac{\beta+\gamma}{z^{\gamma}} f^{\beta}(z)\right)^{\frac{1}{\beta}}$. The operator $A \in H^p$ for $f \in H^p$. Hence $A(B) \in H^{\lambda}$, $\lambda < \frac{1}{2}$ and $I(f) \in H^{\lambda}$, $\lambda < \frac{1}{2}$. From Theorem 7 [1] we obtain the result.

The following result is important for the operator of Singh since β is complex number.

Theorem 5. *If $\beta, \gamma \in \mathbb{C}$, $Re\beta > 0$, $Re\gamma > 0$ and $f, g \in E_{\beta,\gamma}$ with $g'(0) \neq 0$ and g verifies (6) and $f \prec g$ then for the integral operator of Singh I we have:* $I[f] \in H^{\lambda}$, $\lambda < \frac{1}{2}$.

Proof. From Lemma 2 we have for the operator F defined by (5) $f \prec g \Rightarrow F(f) \prec F(g)$ and F is univalent. Hence $F(f) \in H^{\lambda}$, $\lambda < \frac{1}{2}$. $I[f]$ and $F[f]$ are different by a constant. Hence we obtain $I[f] \in H^{\lambda}$, $\lambda < \frac{1}{2}$.

Corollary. If g is an analytic function, $g'(0) \neq 0$ and $Re\left[1 + \frac{zg''(z)}{g'(z)}\right] > -\frac{1}{2}$, then for $G(z) = \frac{1}{2}\int_0^z g(t)\, dt$ we have $G \in H^{\lambda}$ for all $\lambda < \frac{1}{2}$.

REFERENCES

[1] Miclăuş, Gh., *Integral Operator of Singh and Hardy Classes*, Studia Univ. Babeş-Bolyai, Mathematica, 41, 3, (1996), 57-64

[2] Miclăuş, Gh., *Some Integral Operators and Hardy Classes*, Studia Univ. Babeş-Bolyai, Mathematica, 42,2 (1997), 71-77

[3] Miller, S.S., Mocanu, P.T., *The Theory and Applications of Second Order Differential Subordinations*, Studia Univ.Babeş-Bolyai, Math., 34,4 (1989), 3-33

[4] Miller, S.S., Mocanu, Gh., *Differential subordinations and univalent functions*, Michigan Math. J., 28 (1981), 157-171

Received: 1.06. 2002

North University of Baia Mare
Department of Mathematics and Computer Science
Victoriei 76, 4800 Baia Mare, Romania
E-mail: miclaus5@yahoo.com