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FISHER INFORMATION AND TRUNCATED GAMMA DISTRIBUTION

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Abstract. The Fisher information measure is well known in estimation theory. The objective of this paper is to give some definitions and some properties for the truncated Gamma distribution. Also, we shall investigate some measures of the information of the unknown parameters which appear in a such distribution.

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1. Definitions and properties for the truncated Gamma distributions

Let X be a continuous random variable, defined on a probability space (Ω, K, P) and $f(x;\theta)$ its probability density function, where $\theta = (\theta_1, \theta_2)$, θ - a vector parameters (an 2-dimensional parameter), $\theta \in D_\theta$, D_θ *−the parameter space* or the set of admissible values of θ, $D_\theta \subseteq R^2$.

{efinition 1.1. The continuous random variable X follows **the Gamma distribution** with parameters $\theta_1 = a > 0$ and $\theta_1 = b > 0$ if its probability density function is of the following form

$$f_X(x;a,b) = \begin{cases} 0 & if \quad x \le 0 \\ \frac{b^a}{\Gamma(a)} x^{a-1} e^{-bx} & if \quad x > 0, \end{cases} \tag{1.1}$$

where $\Gamma(a) = \int\limits_0^\infty t^{a-1} e^{-t} dt, a > 0$, is **Euler's Gamma function.**

For a such random variable we have $E(X) = \frac{a}{b}$, $Var\ X = \frac{a}{b^2}$.

Definition 1.2. [5] We say that the continuous random variable X has a **Gamma distribution, truncated to the left** at $X = \alpha$ **and to the right** at $X = \beta$, if its probability density function, denoted by $f_{\alpha \leftrightarrow \beta}$, is of the form

$$f_{\alpha\leftrightarrow\beta}(x;a,b) = \begin{cases} C(a,b)x^{a-1}e^{-bx} & if \quad \alpha \le x \le \beta \\ 0 & if \quad x < \alpha\ or >>> \beta \end{cases}, \alpha, \beta \in R, \alpha \ge 0, \tag{1.2}$$

where $C(a,b)$ is a constant with one of the following forms:

$$C(a,b) = \frac{b^a}{\Gamma(a)\left[\Gamma_\beta(a) - \Gamma_\alpha(a)\right]}, \alpha, \beta \in R, 0 \le \alpha \le \beta, \tag{1.3}$$

if $a, b \in R^+$, or

$$C(a,b) = \frac{b^a}{\sum_{k=0}^{a-1} A_{a-1}^k [(\alpha b)^{a-1-k} e^{-\alpha b} - (\beta b)^{a-1-k} e^{-\beta b}]}, \tag{1.4}$$

where

$$A_{a-1}^k = (a-1)(a-2)...[(a-1)-(k-1)],\ k = \overline{0, a-1}, \tag{1.5}$$

if $a \in N^*$, $b \in R^+$ and

$$\Gamma_\beta(a) = \frac{b^a}{\Gamma(a)} \int_0^\beta x^{a-1} e^{-bx} dx, \quad \Gamma_\alpha(a) = \frac{b^a}{\Gamma(a)} \int_0^\alpha x^{a-1} e^{-bx} dx. \tag{1.5a}$$

Lemma 1.1. The family of Gamma distribution, truncated to the left at $X = \alpha$ and to the right at $X = \beta$, $\{f_{\alpha \leftrightarrow \beta}(x; a, b); a, b \in R^+\}$, is an exponential family, that is, we have

$$f_{\alpha \leftrightarrow \beta}(x; a, b) = \frac{b^a}{\Gamma(a)[\Gamma_\beta(a) - \Gamma_\alpha(a)]} exp\left\{\sum_{i=1}^{2} \pi_i(\theta) t_i(x)\right\}, \tag{1.6}$$

if we have in view the definition for an exponential family of distributions.

Theorem 1.1. [5] *If $X_{\alpha \leftrightarrow \beta}$ is a random variable with a Gamma distribution, truncated to the left at $X = \alpha$ and to the right at $X = \beta$, where $\alpha, \beta \in R, \alpha \geq 0$, and its probability density has the form*

$$f_{\alpha \leftrightarrow \beta}(x; a, b) = \begin{cases} \frac{b^a}{\Gamma(a)[\Gamma_\beta(a) - \Gamma_\alpha(a)]} x^{a-1} e^{-bx} & if \quad \alpha \leq x \leq \beta \\ 0 & if \quad x < \alpha \ or \ > \beta \end{cases} \tag{1.7}$$

then the mean value (the expected value) has one of the following forms:

$$E(X_{\alpha \leftrightarrow \beta}) = \frac{a}{b} + \frac{(\alpha b)^a e^{-\alpha b} - (\beta b)^a e^{-\beta b}}{b\Gamma(a)[\Gamma_\beta(a) - \Gamma_\alpha(a)]}, \tag{1.8}$$

if $\alpha, \beta \in R, 0 \leq \alpha \leq \beta$, and $a, b \in R^+$, or

$$E(X_{\alpha \leftrightarrow \beta}) = \frac{a}{b} + \frac{(\alpha b)^a e^{-\alpha b} - (\beta b)^a e^{-\beta b}}{b \sum_{k=0}^{a-1} A_{a-1}^k [(\alpha b)^{a-1-k} e^{-\alpha b} - (\beta b)^{a-1-k} e^{-\beta b}]}, \tag{1.9}$$

if $\alpha, \beta \in R, 0 \leq \alpha \leq \beta$, and $a \in N^*$, $b \in R^+$.

Theorem 1.2. [5] *If $X_{\alpha \leftrightarrow \beta}$ is a random variable with a Gamma distribution, truncated to the left at $X = \alpha$ and to the right at $X = \beta$, where $\alpha, \beta \in R, \alpha \geq 0$, and its* *probability density has the form (1.7), then for the 2 − th order moment* $\alpha_2 = E(X^2_{\alpha\to\beta})$ *and for the variance* $Var(X_{\alpha\to\beta})$*, we have the following expressions*

$$E\left[(X_{\alpha\to\beta})^2\right] = \frac{a(a+1)}{b^2} + \frac{(\alpha b)^{a+1}e^{-\alpha b} - (\beta b)^{a+1}e^{-\beta b}}{b^2\Gamma(a)\left[\Gamma_\beta(a) - \Gamma_\alpha(a)\right]} - \frac{a+1}{b^2}\frac{(\beta b)^a e^{-\beta b} - (\alpha b)^a e^{-\alpha b}}{\Gamma(a)\left[\Gamma_\beta(a) - \Gamma_\alpha(a)\right]} \tag{1.10}$$

$$Var(X_{\alpha\to\beta}) = \frac{a}{b^2} - \frac{(\beta b)^{a+1}e^{-\beta b} - (\alpha b)^{a+1}e^{-\alpha b}}{b^2\Gamma(a)\left[\Gamma_\beta(a) - \Gamma_\alpha(a)\right]} + \frac{a+1}{b^2}\frac{(\beta b)^a e^{-\beta b} - (\alpha b)^a e^{-\alpha b}}{\Gamma(a)\left[\Gamma_\beta(a) - \Gamma_\alpha(a)\right]} - \left[\frac{(\beta b)^a e^{-\beta b} - (\alpha b)^a e^{-\alpha b}}{b\Gamma(a)\left[\Gamma_\beta(a) - \Gamma_\alpha(a)\right]}\right]^2, \tag{1.11}$$

if $\alpha, \beta \in R, 0 \le \alpha \le \beta$*, and* $a, b \in R^+$.

2. The amount of information contained in a random sample

Let C be a statistical population and X a common property for all elements of this population. We suppose that this common property is a continuous random variable which has a probability density function $f(x;\theta)$, where θ is a real *one-dimensional* (or *k-dimensional*) parameter having values in a *parameter space* $D_\theta \subseteq R$ (or $D_\theta \subseteq R^k$).

Let $S_n(X) = (X_1, X_2, ..., X_n)$ denote a random sample of size n from the population C which has the probability density function $f(x;\theta)$, $\theta \in D_\theta$. Our problem is that of defining a statistic $\widehat{\theta} = g(X_1, X_2, ..., X_n)$ so that if $x_1, x_2, ..., x_n$ are the observed experimental values of the sample random variables $X_1, ..., X_n$, then the number $g(x_1, x_2, ..., x_n)$ will be referred to as an **estimate** of θ and is usually written as $\theta_0 = g(x_1, x_2, ..., x_n)$, that is,

$$g : R^n \to R, \tag{2.1}$$

where R^n is the space sample. In this case the statistic $g(X_1, X_2, ..., X_n)$ represents an estimator for the unknown parameter θ.

More, if $\widehat{\theta} = g(X_1, X_2, ...X_n)$ is an unbiased estimator of the unknown parameter θ, then a lower bound of the variance $Var(\widehat{\theta})$ satisfies the inequality

$$Var(\widehat{\theta}) \ge \frac{1}{I_n(\theta)} = \frac{1}{nI_X(\theta)}, \tag{2.2}$$

provided that some regularity conditions concerning the probability density function $f(x;\theta), \theta \in D_\theta$ are satisfied.

Definition 2.1. The quantity $I_X(\theta)$, defined by the relation

$$I_X(\theta) = \int_{\Omega} \left(\frac{\partial \ln f(x;\theta)}{\partial \theta} \right)^2 f(x;\theta)dx. \tag{2.3}$$

is known as **Fisher's information measure** associated with a continuous random variable X.

3. Fisher's information measures for the truncated gamma distribution

Let $X_{\alpha\leftrightarrow\beta}$ be a random variable which has a Gamma distribution, truncated to the left at $X = \alpha$ and to the right at $X = \beta$, with probability density function of the form

$$f_{\alpha\leftrightarrow\beta}(x;a,b) = \begin{cases} C(a,b)x^{a-1}e^{-bx} & if \quad \alpha \leq x \leq \beta \\ 0 & if \quad x < \alpha \ or >>> \beta \end{cases}, \alpha, \beta \in R, \alpha \geq 0, \tag{3.1}$$

where

$$C(a,b) = \frac{b^a}{\Gamma(a)\left[\Gamma_\beta(a) - \Gamma_\alpha(a)\right]}, \alpha, \beta \in R, 0 \leq \alpha \leq \beta, \tag{3.2}$$

if $a, b \in R^+$.

Theorem 3.1. *If $X_{\alpha\leftrightarrow\beta}$ follows a gamma distribution, truncated to the left at $X = \alpha$ and to the right at $X = \beta$, with probability density function of the form (3.1), where $a, b \in R^+$, $a-$ parameter known, $b-$ parameter unknown, then the Fisher information measure corresponding to $X_{\alpha\leftrightarrow\beta}$ has the following form*

$$\begin{aligned} I_{X_{\alpha\leftrightarrow\beta}}(b) = & \frac{a}{b^2} - \frac{(\beta b)^{a+1}e^{-\beta b} - (\alpha b)^{a+1}e^{-\alpha b}}{b^2\Gamma(a)\left[\Gamma_\beta(a) - \Gamma_\alpha(a)\right]} + \\ + & \frac{a-1}{b^2}\frac{(\beta b)^a e^{-\beta b} - (\alpha b)^a e^{-\alpha b}}{\Gamma(a)\left[\Gamma_\beta(a) - \Gamma_\alpha(a)\right]} - \\ - & \left[\frac{(\beta b)^a e^{-\beta b} - (\alpha b)^a e^{-\alpha b}}{b\Gamma(a)\left[\Gamma_\beta(a) - \Gamma_\alpha(a)\right]}\right]^2 \end{aligned} \tag{3.3}$$

Proof. Because $X_{\alpha\leftrightarrow\beta}$ is a continuous random variable and $\theta = b$ is an **unknown parameter** it follows that the Fisher information measure, with respect to the unknown parameter b, has the form

$$\begin{aligned} I_{X_{\alpha\leftrightarrow\beta}}(b) & = \int_\alpha^\beta \left(\frac{\partial \ln f_{\alpha\leftrightarrow\beta}(x;a,b)}{\partial b} \right)^2 f_{\alpha\leftrightarrow\beta}(x;a,b)dx = \\ & = -\int_\alpha^\beta \frac{\partial^2 \ln f_{\alpha\leftrightarrow\beta}(x;a,b)}{\partial b^2} f_{\alpha\leftrightarrow\beta}(x;a,b)dx. \end{aligned} \tag{3.4}$$

Now, by means of the probability density function (3.1), we obtain

$$\begin{aligned} \ln f_{\alpha\leftrightarrow\beta}(x;a,b) & = a\ln b - \ln\Gamma(a) - \ln[\Gamma_\beta(a) - \Gamma_\alpha(a)] + \\ & + (a-1)\ln x - bx, \end{aligned} \tag{3.5}$$

where $\Gamma_\beta(a)$ and $\Gamma_\alpha(a)$ have been specified in (1.5a).

From (3.5), we find that

$$\frac{\partial \ln f_{\alpha\leftrightarrow\beta}(x;a,b)}{\partial b} = \frac{a}{b} - x - \frac{\frac{\partial\Gamma_\beta(a)}{\partial b} - \frac{\partial\Gamma_\alpha(a)}{\partial b}}{\Gamma_\beta(a) - \Gamma_\alpha(a)}. \tag{3.6}$$

Using (1.5a), we get

$$\frac{\partial\, \Gamma_\beta(a)}{\partial b} = \frac{a}{b}[\Gamma_\beta(a) - \Gamma_\beta(a+1)] \tag{3.7}$$

But, for the integral

$$\Gamma_\beta(a+1) = \frac{b^{a+1}}{\Gamma(a+1)} \int_0^\beta x^a e^{-bx} dx, \tag{3.8}$$

then when we apply the well known formula for integration by parts,we obtain a new form

$$\Gamma_\beta(a+1) \;\; = \;\; \Gamma_\beta(a) - \tfrac{(\beta b)^a}{\Gamma(a+1)} e^{-\beta b}. \tag{3.8a}$$

A such relation is holds and in the general case, namely

$$\Gamma_\beta(a+k) \;\; = \;\; \Gamma_b(a+k-1) - \tfrac{(\beta b)^{a+k-1} e^{-\beta b}}{\Gamma(a+k)} \;, k \in N^*. \tag{3.9}$$

From (3.7) and (3.8a), we get the following relation

$$\frac{\partial\, \Gamma_\beta(a)}{\partial b} = \frac{(\beta b)^a}{b\Gamma(a)} e^{-\beta b} \tag{3.10}$$

which is holds and in general case, namely

$$\frac{\partial\, \Gamma_\beta(a+k)}{\partial b} = \frac{a+k}{b} \frac{(\beta b)^{a+k}}{\Gamma(a+k+1)} e^{-\beta b}, k \in N, a+k+1 > 0. \tag{3.11}$$

In a similar manner, from (1.5a), we obtain

$$\frac{\partial\, \Gamma_\alpha(a)}{\partial b} = \frac{(\alpha b)^a}{b\Gamma(a)} e^{-\alpha b} \tag{3.12}$$

From (3.6), (3.10) and (3.12), we conclude that

$$\frac{\partial \ln f_{\alpha\leftrightarrow\beta}(x;a,b)}{\partial b} = \frac{a}{b} - x - \frac{(\beta b)^a e^{-\beta b} - (\alpha b)^a e^{-\alpha b}}{b\Gamma(a)[\Gamma_\beta(a) - \Gamma_\alpha(a)]}. \tag{3.13}$$

Then, from (3.13), we obtain

$$\frac{\partial^2 \ln f_{\alpha\leftrightarrow\beta}(x;a,b)}{\partial b^2} = -\frac{a}{b^2} - \frac{\partial}{\partial b}\left\{\frac{(\beta b)^a e^{-\beta b} - (\alpha b)^a e^{-\alpha b}}{b\Gamma(a)[\Gamma_\beta(a) - \Gamma_\alpha(a)]}\right\} =$$

$$= -\frac{a}{b^2} - \frac{a-1}{b^2}\frac{(\beta b)^a e^{-\beta b} - (\alpha b)^a e^{-\alpha b}}{\Gamma(a)[\Gamma_\beta(a) - \Gamma_\alpha(a)]} + \frac{1}{b^2}\frac{(\beta b)^{a+1} e^{-\beta b} - (\alpha b)^{a+1} e^{-\alpha b}}{\Gamma(a)[\Gamma_\beta(a) - \Gamma_\alpha(a)]} +$$

$$+\left\{\frac{(\beta b)^a e^{-\beta b} - (\alpha b)^a e^{-\alpha b}}{b\Gamma(a)[\Gamma_\beta(a) - \Gamma_\alpha(a)]}\right\}^2 \qquad (3.14)$$

Using this last relation and taking into account (3.4) we can express Fisher's information measure just in the form (3.3). Thus, the proof is complete.

Corollary 3.1. If X follows a Gamma distribution and $X_{\alpha\leftrightarrow\beta}$ follows a Gamma distribution, truncated to the left at $X = \alpha$ and to the right at $X = \beta$, then the Fisher information $I_{X_{\alpha\leftrightarrow\beta}}(b)$ and $Var(X_{\alpha\leftrightarrow\beta})$ always are equal, that is, we have

$$I_{X_{\alpha\leftrightarrow\beta}}(b) = Var(X_{\alpha\leftrightarrow\beta}), \alpha, \beta \in R, 0 \leq \alpha \leq \beta, a, b \in R^+. \qquad (3.15)$$

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