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ABOUT NOMOGRAPHIC REPRESENTATION OF THE PSEUDO-SUM FUNCTIONS

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Abstract. The author provides a study about the nomographic representation of the pseudo-sums with two, three or n terms. It is shown that these pseudo-sums are nomographically represented by plane nomograms with alignment points, by nomograms in space R^3 with coplanar points and by compound nomograms.

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1.Introduction

One of the important problems in nomography consists in finding the conditions of the nomographic representation of the equations (and functions). In many cases these conditions are expressed by the equations with partial derivatives of at least first order of the functions which appear in these equations. However the conditions expressed by equations with partial derivatives are too restrictive and they are not imposed by the nomographic necessities. Only condition of continuity and monotony are, in some cases, sufficient for such nomographic representation. The avoidance of the derivability conditions has led to the "more simple" characterization of the nomographic representation conditions, i.e. the characterization by functional equations. Subsequent to the appearance of the papers of J. Aczél [1]-[3], W. Blaschke [5], F. Radó [8], [9] a connection was made between nomography and functional equations. In this respect a new definition of the nomogram become necessary. It was formulated by F. Radó [8] who used the definitions of Blaschke-Bol [5]. A family of marked curves is a topological image of a fascicle of parallel straight lines. A system of two (marked) curves forms a (marked) network of curves if each curve in the first family crosses the curve in the other family in not more than one point. Three families of (marked) curves which considered two by two forms net curves make up a (marked) tissue.

Definition 1. The nomogram with marked lines is a plane marked tissue.

By noting the marks of the curves of the families by z_1, z_2, z_3, the value

$$z_3 = f(z_1, z_2) \tag{1}$$

corresponds to the pair (z_1, z_2). From the above definitions it results that the function f is continuous and monotonic. Then we have the theorem [8]. *The equation (1) can be nomographically represented by a nomogram with marked lines if and only if the function*

$f : D_1' \subset R^2 \to R$ *is a continuous and strictly monotone one (with respect to any variable).*
Let us consider the equation

$$X(z_1, z_2, z_3) = 0 \tag{2}$$

where $X : D_1 \subset R^3 \to R$ is a continuous function that defines each of its variables as an implicit function of the other two variables. The problem (known as the anamorphisis) is of building a nomogram with marked lines for the equation (2) (or (1)) by arbitrarily choosing the network of curves z_1, z_2 as a straight line network.

Theorem 1. [8] *The necessary and sufficient condition so that the equation (1) (where f is continuous and strictly monotonic) be nomographically represented by a nomogram with marked straight lines is that it could be written in the form*

$$\begin{vmatrix} f_1(z_1) & g_1(z_1) & h_1(z_1) \\ f_2(z_2) & g_2(z_2) & h_2(z_2) \\ f_3(z_3) & g_3(z_3) & h_3(z_3) \end{vmatrix} = 0 \tag{3}$$

where $f_i(z_i), g_i(z_i), h_i(z_i)$, $i = \overline{1,3}$ are continuous functions.

Taking into account the fact that by a correlation of the nomogram's plane and by conveniently choosing the projective system of coordinates, the marked family of curves is transformed into a marked scale, we notice that a nomogram with alignment points, with three scales, is the dual image of a nomogram with three marked straight lines.

This paper will approach the nomographic representation of the pseudo-sums of two, three or more variables. Mention will also be made of the functional equations that have as a solution the pseudo-sums.

2. The functions of two variables which are pseudo-sums

a) The characterization of the pseudo-sums with two terms. The following canonical forms are known for the equations with three variables of the third nomographical order, which can be represented nomographically by nomograms with alignment points of genus zero

$$F(z_1) + G(z_2) = H(z_3); \qquad F(z_1)G(z_2)H(z_3) = 1 \tag{4;5}$$

$F(z_1)G(z_2)H(z_3) = 1$where F, G, H are the real functions of one variable, continuous and straight monotonic.Thus the equation (4) contains three functions of one variable, F, G, H, that are continuous and strictly monotonic. The inverse functions F^{-1}, G^{-1}, H^{-1} are also continuous and strictly monotonic. The equation (4) can still be written under the form

$$z_3 = f(z_1, z_2) = H^{-1}[F(z_1) + G(z_2)]. \tag{6}$$

Definition 2. The function of two variables $z_3 = f(z_1, z_2)$, defined in a plane domain, is named *pseudo-sum with two terms* if the relation (6) is fulfiled in each point of the domain and the functions F, G, H are continuous and strictly monotonic [8].

Theorem 2. *The necessary and sufficient condition so that the continuous and strictly monotonous function of two variables, $f(z_1, z_2)$, be pseudo-sum with two terms is there exists the decomposition (6).*

J. Aczél [1] has characterized the pseudo-sums of the form

$$z_3 = f(z_1, z_2) = H^{-1}[aH(z_1) + bH(z_2) + c] \tag{7}$$

(where H is continuous and strictly monotonic), by the functional equation of the bisymmetry

$$f[f(u, z_1), f(z_2, v)] = f[f(u, z_2), f(z_1, v)]. \tag{8}$$

Here f is a continuous and strictly monotonic function in a domain in R^2 space.We notice that the functions of form (7) is a subclass of the functions (6). Next Aczél found (in additional conditions of reflexivity and symmetry imposed on the function f) the solutions, of the bisymmetrical equation (8), expressed in the form

$$z_3 = f(z_1, z_2) = H^{-1}\left[\frac{H(z_1) + H(z_2)}{2}\right] \tag{9}$$

He solved this problem, following the axiomatical characterization of the functions named quasiarithmetical means, proposed by A. Kolmogorov [7]

$$M_n(x_1, x_2, \ldots, x_n) = F^{-1}\left[\frac{F(x_1) + F(x_2) + \ldots + F(x_n)}{n}\right] \tag{10}$$

F. Rado [8] has demonstrated that the pseudo-sums are continuous and strictly monotonic solutions for the functional equation which generalizes the equation (8)

$$f[\overline{f}(u, z_1), \overline{\overline{f}}(z_2, v)] = f[\overline{f}(u, z_2), \overline{\overline{f}}(z_1, v)]. \tag{11}$$

Here the functions $\overline{f}, \overline{\overline{f}}$ are also defined by the equation (1) solved with respect to z_1, respectively z_2 (i.e. $z_1 = \overline{f}(z_2, z_3)$, and $z_2 = \overline{\overline{f}}(z_3, z_1)$).

b) The nomographical representation of the pseudo-sums with two terms

The equations with three variables of the third nomographic order, (4) and (5) are represented by nomograms with alignment points of genus zero, and with three straight line scales. The Soreau forms, with d as a parameter, for these equations are:

$$\begin{vmatrix} 0 & F(z_1) & 1 \\ d & G(z_2) & 1 \\ \frac{d}{2} & \frac{H(z_3)}{2} & 1 \end{vmatrix} = 0 \qquad \begin{vmatrix} d & \frac{1}{F(z_1)} & 1 \\ 0 & G(z_2) & 1 \\ \frac{d}{1-H(z_3)} & 0 & 1 \end{vmatrix} = 0 \tag{12}$$

hence we obtain the equations of the scales of the nomogram. $scale(z_1)$: $x = 0$, $y = F(z_1)$ $scale(z_2)$: $x = d$, $y = G(z_2)$ $scale(z_3)$: $x = \frac{d}{2}$, $y = \frac{H(z_3)}{2}$ respectively $scale(z_1)$: $x = d$, $y = \frac{1}{F(z_1)}$ $scale(z_2)$: $x = 0$, $y = G(z_2)$, $scale(z_3)$: $x = \frac{d}{1-H(z_3)}$, $y = 0$. These scales are situated on the three parallel straight lines and, respectively, on two parallel straight lines and a third that intersects them. If the scales of the nomograms (for (4) and (5)) are situated either on three straight lines which cross each other in a point, or on three straight lines that form a triangle, then we have the Soreau equations

$$\begin{vmatrix} \frac{1}{F(z_1)} & 0 & 1 \\ 0 & \frac{1}{G(z_2)} & 1 \\ \frac{1}{H(z_3)} & \frac{1}{H(z_3)} & 1 \end{vmatrix} = 0 \qquad \begin{vmatrix} 0 & \frac{F(z_1)}{1+F(z_1)} & 1 \\ \frac{1}{1+G(z_2)} & 0 & 1 \\ \frac{H(z_3)}{H(z_3)-1} & \frac{1}{1-H(z_3)} & 1 \end{vmatrix} = 0 \tag{13}$$

from where, by adding the parameters that are necessary in the nomogram construction we can infer the equations for the nomogram scales. So, we can enunciate

Theorem 3. *The plane nomogram with alignment points of genus zero corresponds to the equation (4) (respectively to the pseudo-sum of two variables with two terms (6)); reciprocally pseudo-sum of two variables with two terms (6) (or any equation (4)) is nomographically represented by a nomogram of genus zero.*

Remark 1. Taking into consideration the theorem 3 and the point 2a), we realize that the pseudo-sums (7) (a particular case of (6)) will be nomographically represented by nomograms with alignment points of genus zero with three homotetic straight line scales.The pseudo-sums (9) will also be represented nomographically by nomograms with alignment points, that have the three scales situated on three parallel equidistant straight lines (the scale z_3 is situated in the middle).

3. The pseudo-sums with three terms

a) The characterization of the pseudo-sums with three terms. The functional equation

$$g[\varphi(z_1, z_2), z_3] = h[z_1, \psi(z_2, z_3)] \tag{14}$$

where there exists four unknown functions has been studied. The equation (14) represents a generalization of the associativity equation. Therefore we want to represent the functions of three variables $f(z_1, z_2, z_3)$ as superpositions of functions of one or two variables. So, we have

Theorem 4. [8] *The general continuous and strictly monotonic solutions of the equation (14) are the functions*

$$\begin{aligned} \varphi(z_1, z_2) &= H_1^{-1}[F_1(z_1) + G_1(z_2)]; \;\; \psi(z_1, z_2) = H_2^{-1}[G_1(z_1) + G_2(z_2)] \\ g(z_1, z_2) &= H_3^{-1}[H_1(z_1) + G_2(z_2)]; \;\; h(z_1, z_2) = H_3^{-1}[F_1(z_1) + H_2(z_2)] \end{aligned} \tag{15}$$

where F_1, G_1, G_2, H_1, H_2 are arbitrary but continuous and strictly monotonic functions.

By replacing (15) in (14) and noting

$$f(z_1, z_2, z_3) = H_3^{-1}[F_1(z_1) + G_1(z_2) + G_2(z_3)] \tag{16}$$

or by noting again we obtain

$$f(z_1, z_2, z_3) = K^{-1}[F(z_1) + G(z_2) + H(z_3)] \tag{17}$$

Definition 3. The function of three variables (17) is a *pseudo-sum with three terms* if in every point of a domain from R^3 space there are the relation (17) and the functions F, G, H, K are continuous and strictly monotonic. Therefore we can enunciate the following

Theorem 5. [8] *The necessary and sufficient condition so that the continuous and strictly monotonous functions of three variables be pseudo-sums with three terms is that the decomposition*

$$f(z_1, z_2, z_3) = g[\varphi(z_1, z_2), z_3] = h[z_1, \psi(z_2, z_3)]. \tag{18}$$

should be possible.

b) The nomographical representation of the pseudo-sums with three terms

The notions introduced in section **1, 2** referring to the representation of the equation with three variables (2) and respectively of a function of two variables by a nomogram with marked lines can be further employed in the case of an equation of four variables as well as of a function of three variables. This extention is performed on the purpose of representing them either by a nomogram in space with coplanary points, or by a nomogram in space with families of surfaces. Here we will extend the concept of family of curves, networks and tissue to the three-dimensional space.

A family of surfaces from a domain $E \subset R^3$ is the topological image of a system of parallel planes in a domain $E_1 \subset R^3$. The real constants in the equation of the family of surfaces represent the marks of the family. A system made up of three families of (marked) surfaces in $E \subset R^3$ is called a *(marked) network of surfaces* in $E \subset R^3$, if the three above families together with the domain E are homomorphic with the three families of parallel planes from E_1 (two by two orthogonal). A system of four families of (marked) surfaces in named *(marked) tissue in space* if any group of three families forms a network of surfaces.Let us consider a tissue in space consisting of three families of planes which are parallel with the coordinate planes and a forth family made of the level surfaxes of the function (17), i.e. $f(z_1, z_2, z_3, z_4) = K^{-1}[F(z_1) + G(z_2) + H(z_3)] = const.$

By applying a topological transformation to this tissue so that the families of parallel planes also remain parallel with the coordinate planes $z_1' = F(z_1)$, $z_2' = G(z_2)$, $z_3' = H(z_3)$ we also obtain a family of parallel planes. Thus we have $K^{-1}[z_1' + z_2' + z_3'] = const.$ Such a tissue is names a *regular tissue in space*.

Theorem 6. *Pseudo-sum with three terms is nomographically represented by a regular tissue in space R^3 and reciprocally.*

Applying a correlation to the regular tissue we obtain a nomogram in space with coplanary points, with four straight line parallel scales. We obtain then

Theorem 7. *The pseudo-sum with three terms (17) is nomographically represented by a nomogram in space with coplanary points (with four straight line parallel scales) and reciprocally.*

If the function of three variables (18) (or the equation with four variables)

$$z_4 = f(z_1, z_2, z_3) \tag{19}$$

admits a decomposition of the form (14), then the equation (19) can be represented by a compound nomogram consisting of two nomograms of (marked) lines. Each of these two nomograms contains two families of marked lines and an unmarked family common to both. Noting by

$$\xi = \varphi(z_1, z_2), \quad z_4 = F(\xi, z_3) \tag{20}$$

for each of the equations (20) we can write the Soreau equation

$$\begin{vmatrix} \alpha(\xi) & \beta(\xi) & \gamma(\xi) \\ f_1(z_1) & g_1(z_1) & h_1(z_1) \\ f_2(z_2) & g_2(z_2) & h_2(z_2) \end{vmatrix} = 0; \quad \begin{vmatrix} \alpha(\xi) & \beta(\xi) & \gamma(\xi) \\ f_3(z_3) & g_3(z_3) & h_3(z_3) \\ f_4(z_4) & g_4(z_4) & h_4(z_4) \end{vmatrix} = 0 \tag{21}$$

Taking into consideration the above definitions we find out that to the pseudo-sums with three terms corresponds a compound nomogram by two plane nomograms with alignment points with parallel straight line scales and a "mute" scale.

Theorem 8. *To the equation with four variables (19), where the function f is continuous and strictly monotonous, and more it admits a decomposition in the form (18), correspond a compound plane nomogram, reciprocally a compound nomogram represents nomographically an equation with four variables where one of its variables is an implicite function of other three variables.*

For the pseudo-sums with n terms we can also build the multidimensional nomogram.

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