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APPROXIMATION OF INTEGRABLE FUNCTIONS BY GENERALIZED ABEL-POISSON MEANS

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Abstract. Special representation (Zamansky-type representation) is obtained for generalized Abel-Poisson means of Fourier series for integrable functions. More exactly, the deviation of such the function and its generalized Abel-Poisson means is represented as an improper integral of second order symmetric difference for the function and a remainder. This remainder is estimated from both above and below in terms of second order modulus of smoothness for the function.

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1. Introduction

Let $f \in L_p (1 \leq p \leq \infty)$ be a 2π-periodic function and let

$$f(x) \sim \sum_{k=-\infty}^{\infty} c_k e^{ikx} \equiv \sum_{k=-\infty}^{\infty} A_k(x) \tag{1}$$

be the Fourier series of f with c_k as Fourier coefficients of f with respect to trigonometric system.

Let

$$\Delta_\delta f(x) = f(x+\delta) - f(x-\delta),$$

$$\Delta_\delta^1 f(x) = \Delta_\delta f(x), \quad \Delta_\delta^s f(x) = \Delta(\Delta^{s-1} f(x)),$$

be the symmetric difference for f of correspondent orders with the step δ taken at a point x and

$$\omega_s(f;h) = \sup_{0<\delta\leq h} \|\Delta_\delta^s f(x)\|$$

be the modulus of smoothness of order s for with the step h. The norms here and through the paper are taken in L_p.

Through the paper, we will deal with two different means of (1):

a) Riesz means:

$$R_n^s(f;x) = \sum_{|k|\le n} \left(1 - \left(\frac{|k|}{n+1}\right)^s\right) A_k(x), \quad s > 0$$

(they can be reduced to arithmetic means if $s = 1$ and we will wright $R_n^1 \equiv R_n$); b) generalized Abel-Poisson means in the form:

$$P_n^\alpha(f;x) = \sum_{k=-\infty}^{\infty} e^{-\alpha\frac{|k|}{n+1}} A_k(x), \quad \alpha > 0$$

(these means can be reduced to usual Abel-Poisson means if $\alpha = 1$).

Classical result by M. Zamansky [1] obtained first for a subclass of the class of continuous functions is

$$f(x) - R_n(f;x) = \frac{-1}{2\pi} \int_1^\infty \Delta_{\frac{t}{n+1}} f(x) t^{-2} dt + \tau_n(f;x), \tag{2}$$

where the remainder τ_n satisfies the condition

$$\|\tau_n(f;x)\|_C \le B$$

for some $B > 0$ which is independent of f and n. In this form, the result is obtained in [2].

The results of this kind for different means are known as Zamansky-type results for linear means of (1).

2. Results

In this paper, the Zamansky-type representation is obtained for Abel-Poisson means of (1) and more than that the estimation for the remainder in this representation is bilateral (from both above and below) but not only from above as in classical Zamansky-type result.

Theorem 1. *If $f \in L_p (1 \le p \le \infty)$ with the Fourier series (1) then there are the constants C_1, C_2 which are independent of and n such that*

$$f(x) - P_n^\alpha(f;x) = \alpha(f(x) - R_n^\alpha(f;x)) + \tau_n(f;x),$$

$$C_1\omega_2\left(f; \frac{1}{n+1}\right) \le \|\tau_n(f;x)\| \le C_2\omega_2\left(f; \frac{1}{n+1}\right) \tag{3}$$

Theorem 2. *If f satisfies the conditions of Theorem 1, then there are the constants B_1, B_2, which are independent of f and n such that for $\alpha > 1$*

$$f(x) - P_n^{\alpha}(f;x) = \frac{-\alpha}{2\pi}\int_1^{\infty} \Delta^2_{\frac{t}{n+1}} f(x) t^{-2} dt + \delta_n(f;x),$$

$$B_1\omega_2\left(f;\frac{1}{n+1}\right) \leq \|\delta_n(f;x)\| \leq B_2\omega_2\left(f;\frac{1}{n+1}\right).$$

3. Proofs

To prove these theorems we use the technique developed in the series of our publications starting from [4] till [3] and devoted to Zamansky-type representations in different situations. This technique is based on two basic positions: comparison principle for linear means of Fourier series proposed by R.M. Trigub (see [5]) and then boundedness of correspondent norms of operators (Lebesgue constants).

Say, to prove the double inequality in (3) we will prove first that there are constants C_3, C_4(independent f of and) such that the inequality

$$C_3\left\|f(x) - R_n^2(f;x)\right\| \leq \|\tau_n(f;x)\| \leq C_4\left\|f(x) - R_n^2(f;x)\right\| \qquad (4)$$

holds.

The transitional function in the comparison principle (see [3]) for the right-side inequality in (4) is

$$\Lambda(u) = \begin{cases} -\frac{\alpha}{2}, & u = 0, \\ -\frac{1}{u^2}(1-\alpha) - e^{-\alpha u} + \alpha(1-u), & u \in (-1,0)\cup(0,1), \\ 1 - \alpha - e^{-\alpha u}, & |u| \geq 1 \end{cases}$$

It remains to check the boundedness of Lebesgue constants correspondent to this function. Then it remains to use the double estimation for the deviation

$$\left\|f - R_n^2\right\|. \qquad (5)$$

To prove theorem 2 we will use the representation (2) and Theorem 1.

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