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CARISTI TYPE OPERATORS AND APPLICATIONS

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Abstract. The purpose of this paper is to present some results on multi-valued Caristi type operators between metric spaces. The case of metric spaces endowed with a weak distance, in Kada-Suzuki-Takahashi sense, is considered.

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1. Introduction

Caristi's fixed point theorem states that each operator f from a complete metric space (X, d) into itself satisfying the condition:

there exists a proper lower semicontinuous function $\varphi : X \to \mathbb{R}_+ \cup \{+\infty\}$ such that:

$$d(x, f(x)) + \varphi(f(x)) \leq \varphi(x), \text{ for each } x \in X \tag{1.1}$$

has at least a fixed point $x^* \in X$, i. e. $x^* = f(x^*)$. (see Caristi [2]).

For the multi-valued case, there exist several results involving multi-valued Caristi type conditions. See for example, Mizoguchi-Takahashi [7], Maschler-Peleg [6], van Hot [3], etc.

The purpose of this paper is to present some results on multi-valued Caristi type operators between metric spaces. The case of metric spaces endowed with a weak distance, in Kada-Suzuki-Takahashi sense, is considered.

2 Preliminaries

Throughout this paper (X, d) is a complete metric space, and $F : X \multimap X$ denotes a multi-valued operator.

If (X, d) is a metric space, then $\mathcal{P}(X)$ will denote the space of all subsets of X. Also, we denote by $P(X)$ the space of all nonempty subsets of X and by $P_p(X)$ the set of all nonempty subsets of X having the property "p", where "p" could be: $cl = closed$, $b = bounded$, $cp = compact$, $cv = convex$ (for normed spaces X), etc.

We consider the following (generalized) functionals:

$$D : P(X) \times P(X) \to \mathbb{R}_+,\ D(A,B) = \inf \{ d(a,b) \mid a \in A,\ b \in B \}$$

$$H : P(X)\times P(X) \to \mathbb{R}_+ \cup \{+\infty\},\ H(A,B) = \max \{ \sup_{a\in A} D(a,B), \sup_{b\in B} D(b,A) \}.$$

H is called the Pompeiu-Hausdorff generalized functional and it is well-known that if (X,d) is a complete metric space, then $(P_{cl}(X), H)$ is also a complete metric space.

Definition 1 *Let (X,d) be a metric space. Then a multi-valued operator $F : X \to P(X)$ is called:*

a) (M-T)- Caristi type multifunction if there exists a proper lower semicontinuous function $\varphi : X \to \mathbb{R}_+ \cup \{+\infty\}$ such that

$$\text{for each } x \in X, \text{ there is } y \in F(x) \text{ so that } d(x,y) + \varphi(y) \le \varphi(x)$$

(see Mizoguchi-Takahashi [7])

b) (M-P)- Caristi type multifunction if there exists a proper lower semicontinuous function $\varphi : X \to \mathbb{R}_+ \cup \{+\infty\}$ such that

$$\text{for each } x \in X \text{ and each } y \in F(x) \text{ we have } d(x,y) + \varphi(y) \le \varphi(x).$$

(see Maschler-Peleg [6])

c) (vH)- Caristi type multifunction if F has closed values and there exists a proper lower semicontinuous function $\varphi : X \to \mathbb{R}_+ \cup \{+\infty\}$ such that

$$\text{for each } x \in X,\ \inf \{ d(x,y) + \varphi(y) \ :\ y \in F(x) \} \le \varphi(x)$$

(see van Hot [3])

d) Kannan type multifunction if there exists $a \in [0, \frac{1}{2}[$ such that

$$H(F(x), F(y)) \le a[D(x, F(x)) + D(y, F(y))], \text{ for each } x, y \in X.$$

e) Reich type multifunction if there exist $a, b, c \in \mathbb{R}_+$, with $a + b + c < 1$ such that

$$H(F(x), F(y)) \le ad(x,y) + bD(x, F(x)) + cD(y, F(y)), \text{ for each } x, y \in X.$$

If $b = c = 0$, then F is called a multi-valued a-contraction.

Remark 1 *It is quite obviously that if F satisfies a (M-P)- Caristi type condition then F is a (M-T)- Caristi type multifunction and any (M-T)- Caristi type multifunction satisfies a (vH)- Caristi type condition.*

Definition 2 *Let (X, d) be a metric space and $F : X \to P(X)$ be a multi-valued map. Then an element $x^* \in X$ is called a fixed point of F if $x^* \in F(x^*)$. We denote by $Fix(F)$ the fixed points set of F.*

In 1996, Kada, Suzuki and Takahashi introduced the concept of w-distance on a metric space as follows.

Definition 3 *Let (X, d) be a metric space. Then a function $p : X \times X \to \mathbb{R}_+$ is called a w-distance on X if the following are satisfied:*

(1) $p(x, z) \leq p(x, y) + p(y, z)$, for any $x, y, z \in X$

(2) for any $x \in X$, $p(x, \cdot) : X \to \mathbb{R}_+$ is lower semicontinuous

(3) for any $\varepsilon > 0$, there exists $\delta > 0$ such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ imply $d(x, y) \leq \varepsilon$.

Some properties of the w-distance are contained in:

Lemma 1 *Let (X, d) be a metric space and p be a w-distance on X. Let $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}$ be sequences in X, let $(\alpha_n)_{n\in\mathbb{N}}, (\beta_n)_{n\in\mathbb{N}}$ be sequences in $\mathbb{R}_+$ converging to 0 and let $y, z \in X$. Then the following hold:*

(i) if $p(x_n, x_m) \leq \alpha_n$, for any $n, m \in \mathbb{N}$ with $m > n$, then (x_n) is a Cauchy sequence.

(ii) if $p(x_n, y_n) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$, for any $n \in \mathbb{N}$ then (y_n) converges to z.

For examples and related results, see Kada, Suzuki and Takahashi [4].

3 Multi-valued Caristi type operators

It was proved by L. van Hot that any multi-valued a-contraction F on a metric space X is a (vH)-Caristi type multi-function with a function $\varphi : X \to \mathbb{R}_+$, $\varphi(x) = \frac{1}{1-a}D(x, F(x))$. Moreover, if is a multi-valued a-contraction with nonempty and compact values then F satisfies a (M-T)- Caristi type condition with a same function $\varphi(x) = \frac{1}{1-a}D(x, F(x))$.

Let us remark now, that any Reich type multi-function (and hence in particular any Kannan multi-function) is a (vH)- Caristi type multi-function with a function φ given by $\varphi(x) = \frac{1-c}{1-a-b-c}D(x, F(x))$.

Definition 4 *If (X,d) is a metric space, then a multi-valued operator $F : X \to P(X)$ is said to be a Reich type graphic contraction if there exist $a, b, c \in \mathbb{R}_+$, with $a+b+c<1$ such that*

$$H(F(x), F(y)) \leq ad(x,y) + bD(x, F(x)) + cD(y, F(y)),$$

for each $x \in X$ and each $y \in F(x)$.

A connection between multi-valued Reich type graphic contractions and multi-valued Caristi type operators is given in:

Lemma 2 *Let (X,d) be a metric space and let $F : X \to P(X)$ be a Reich type graphic contraction. Then F is a (vH)- Caristi type multi-function.*

For the case of complete metric spaces endowed with a w-distance the following generalization of the Covitz-Nadler fixed point principle for multi-functions was proved by Suzuki and Takahashi in [11]. We need, first, a definition.

Definition 5 *Let (X,d) be a metric space. A multi-valued mapping $F : X \to P(X)$ is called p-contractive if there exist a w-distance p on X and a real number $a \in [0,1[$ such that for any $x_1, x_2 \in X$ and each $y_1 \in F(x_1)$ there exists $y_2 \in F(x_2)$ so that $p(y_1, y_2) \leq ap(x_1, x_2)$.*

Theorem 3 *Let (X,d) be a complete metric space and $\varphi : X \to \mathbb{R}_+ \cup \{+\infty\}$ be a proper lower semicontinuous function. Let $F : X \to P_{cl}(X)$ be a p-contractive multi-function. Then there exists $x^* \in X$ a fixed point for F and $p(x^*, x^*) = 0$.*

Some extensions of the previous result are:

Theorem 4 *Let (X,d) be a complete metric space and $F : X \to P_{cl}(X)$ be a closed multi-valued operator such that the following assumption holds:*

there exist a w-distance p on X and a real number $a \in [0,1[$ so that for any $x \in X$ and any $y_1 \in F(x)$ there is $y_2 \in F(y_1)$ such that $p(y_1, y_2) \leq ap(x, y_1)$.

Then there exists $x^* \in X$ a fixed point for F and $p(x^*, x^*) = 0$.

Theorem 5 *Let (X,d) be a complete metric space and $\varphi : X \to \mathbb{R}_+ \cup \{+\infty\}$ be a proper lower semicontinuous function. Let $F : X \to P_{cl}(X)$ be a closed multi-valued operator having the following property:*

there exists a w-distance p on X so that for each $x \in X$ there is $y \in F(x)$ we have $p(x,y) + \varphi(y) \leq \varphi(x)$.

Then $FixF \neq \emptyset$.

Moreover, if F satisfies the stronger condition:

there exists a w-distance p on X so that for each $x \in X$ *and for each* $y \in F(x)$ *we have* $p(x,y) + \varphi(y) \leq \varphi(x)$,

then there exists $x^* \in X$ *a fixed point for T and* $p(x^*, x^*) = 0$.

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