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SOME EMBEDDINGS THEOREMS FOR (n,2)-RINGS

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Abstract. In this paper we generalize two embeding theorems for rings. Any commutative $(n,2)-$semiring can be embeded isomorphically in a commutative $(n,2)-$ring and every $(n,2)-$ring R can be embeded in the $(n,2)-$ring of square matrices of order n with elements from R.

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1. Introduction

The purpose of this paper is to give a construction for $(n,2)$-rings of quotient of a commutative $(n,2)$-semiring, which generalizes a well known theorem: any commutative semiring can be embeded isomorphically in to a commutative ring. An application of the classical theorem is the extension of the semiring of natural numbers to the ring of integers.

We use the standard notions of n-semigroups, n-groups, $(n,m)-$semirings and (n,m)-rings, as they appear, for example, in [3], [8], [1].

Let R be a set. Throught the paper we will make use of notations: we will ofter write a_1^n instead $a_1, a_2, ..., a_n$; if k consecutive terms coincide we use the short notation $\overset{(k)}{a}$. A set R togheter with an n-ary operation $()_\circ : R^n \to R$ is called n-semigroup if for any $k \in \{1, 2, \ldots, n-1\}$ and all $a_1, a_2, \ldots, a_{2n-1} \in R$ the following associativity laws hold:$\left((a_1^n)_\circ, a_{n+1}^{2n-1}\right)_\circ = \left(a_1^k \left(a_{k+1}^{k+n}\right)_\circ a_{k+n+1}^{2n-1}\right)_\circ$

An n-group $(R, ()_\circ)$ is an n-semigroup in which the equations $\left(a_1^{i-1}, x, a_{i+1}^n\right)_\circ = a_i$ have a unique solution in R for arbitrary $a_1, \ldots, a_n \in R$ and for each $i \in \{1, 2, \ldots n\}$.

An n-semigroup (n-group) $(R, ()_\circ)$ is called semicommutative (commutative) if $\left(a_1, a_2^{n-1}, a_n\right)_\circ = \left(a_n, a_2^{n-1}, a_1\right)_\circ$ (respectively $(a_1^n)_\circ = (a_{\sigma(1)}^{\sigma(n)})_\circ$) for any set of elements $a_1, \ldots, a_n \in R$ (for any permutation $\sigma \in S_n$).

An n-semigroup $(R, ()_\circ)$ is called i-cancellative with respect to $S \subseteq R$, $i \in \{1, \ldots, n\}$, if for all $s_j \in S$, $j \in \{1, \ldots, n\} \setminus \{i\}$, $a, b \in R$, the following implication holds: $\left(s_1^{i-1}, a, s_{i+1}^n\right)_\circ = \left(s_1^{i-1}, b, s_{i+1}^n\right)_\circ \Rightarrow a = b$, and cancellative with respect to S, if it is i-cancellative for every $i \in \{1, 2, \ldots n\}$.

A sequence a_1^{n-1} is called right (left) unit in the n-semigroup $(R,()_\circ)$ if for any $x \in R$ we have $(x, a_1^{n-1})_\circ = x$ ($(a_1^{n-1}, x)_\circ = x$). In an n-group $(R,()_\circ)$ the unique solution of the equation $\left(\overset{(n-1)}{a}\ x\right)_\circ = a$ is called the querelement of a and is denoted by $\overline{a}$. In an n-group the sequence $\overset{(i-1)}{a}, \overline{a}, \overset{(n-i-1)}{a}$ is a right ant left unit for any $i \in \{1, 2, \ldots, n-1\}$.

In an n-semigroup $(R,()_\circ)$ an element $e \in R$ is called a neutral element if $\left(\overset{(i-1)}{e}\ a\ \overset{(n-i)}{e}\right)_\circ = a$, for all $a \in R$ and $i = 1, 2, \ldots, n$.

An algebraic structure $(R,()_\circ,\cdot)$ is called an $(n,2)$-semiring $n \geq 2$ if $(R,()_\circ)$ is a commutative and cancellative n-semigroup, $(R,\cdot)$ is a binary semigroup and the multiplication is both left and right distributive with respect to n-ary addition $()_\circ$. A neutral element of $(R,\cdot)$ (if it exists) is called an identity. If $(R,()_\circ)$ is a commutative n-group then an $(n,2)$-semiring $(R,()_\circ,\cdot)$ is called an $(n,2)$-ring [1].

An $(n,2)$-ring (semiring) is called commutative or cancellative if the semigroup $(R,\cdot)$ has that property. In an $(n,2)$-ring the following relations hold: $\overline{(a_1^n)_\circ} = (\overline{a_1^n})_\circ$ and $\overline{a \cdot b} = \overline{a} \cdot b = a \cdot \overline{b}$ for all $a_i, a, b \in R$, $i = 1, 2, \ldots, n$.

Example 1. The set of natural numbers $\mathbb{N}$ togheter with an n-ary operation

$$()_\circ : \mathbb{N}^n \to \mathbb{N}; \qquad (k_1^n)_\circ = \sum_{i=1}^n k_i + 1$$

and a binary operation

$$* : \mathbb{N}^2 \to \mathbb{N}; k_1 * k_2 = (n-1)k_1k_2 + k_1 + k_2$$

is a commutative and cancellative $(n,2)$-semiring with identity element $0 \in \mathbb{N}$ and without zero element.

Example 2. The set of integers $\mathbb{Z}$ togheter the above operations is a commutative and cancellative $(n,2)$-ring because for all $k \in \mathbb{Z}$ there is the querelement $\overline{k} = (2-n)k - 1 \in \mathbb{Z}$.

Example 3. Let $\mathcal{M}_m(R) = \{A = (a_{ij})_{1 \leq i,j \leq m} \,|\, a_{ij} \in R; 1 \leq i,j \leq m\}$ $m \equiv 1 (\mathrm{mod}\, n-1)$ be the set of all square matrices of the order m with elements of $(n,2)$-ring $(R,()_\circ,\cdot)$. We define an n-ary and a binary operations such that:

If $A_k = (a_{ij}^k)_{1 \leq i,j \leq m}$, $k = 1, 2, \ldots, n$,

$$[A_1^n]_+ = (b_{ij})_{1 \leq i,j \leq m}, \qquad b_{ij} = (a_{ij}^1, a_{ij}^2, \ldots, a_{ij}^n)_\circ.$$

If $A = (a_{ij})_{1 \leq i,j \leq m}$ and $C = (c_{ij})_{1 \leq i,j \leq m}$

$$A * C = (d_{ij})_{1 \leq i,j \leq m}, \qquad d_{ij} = (\ldots(a_{i1}c_{1j}, a_{i2}c_{2j}, \ldots, a_{in}c_{nj})_\circ, \ldots, a_{im}c_{mj})_\circ.$$

It is easy to proof that $(\mathcal{M}_m(R), []_+, *)$ is an $(n,2)$-ring.

2. The main results

Theorem 1. *Any commutative $(n,2)$-semiring can be embeded in a commutative $(n,2)-$ring.*

Proof. Let $(R,()_{\circ},\cdot)$ be a commutative $(n,2)-$semiring. As in [7], [5] we define an equivalence "$\sim$" on R^n by $as_2^n \sim bt_2^n \Leftrightarrow (a,t_2^n)_{\circ} = (b,s_2^n)_{\circ}$. The equivalence class of the $n-$uple as_2^n is denoted by $\frac{a}{s_2^n}$, whille the factor-set $R^n/\sim$ is denoted $R_{R^{n-1}}$. As consequeces of the above definition note that

$$\frac{a}{s_2^n} = \frac{(at_2^n)_{\circ}}{(s_2^n t_2)_{\circ} t_3^n}, \ \forall t_2^n \in R$$

We define the n-ary operation $()_+ : (R_{R^{n-1}})^n \to R_{R^{n-1}}$ by

$$\left(\frac{a_1}{s_{12}^{1n}},\dots,\frac{a_n}{s_{n2}^{nn}}\right)_+ = \frac{(a_1^n)_{\circ}}{(s_{12}^{n2})_{\circ},(s_{13}^{n3})_{\circ},\dots(s_{1n}^{nn})_{\circ}}$$

and a binary operation $*: (R_{R^{n-1}})^2 \to R_{R^{n-1}}$ by

$$\frac{a}{s_2^n} * \frac{b}{t_2^n} = \frac{((\dots((ab,s_2t_2,\dots,s_2t_n)_{\circ},s_3t_2,\dots,s_3t_n)_{\circ},\dots)_{\circ},s_nt_2,\dots,s_nt_n)_{\circ}}{(at_2,\dots,at_n,bs_2)_{\circ},bs_3,\dots,bs_n}.$$

It is easy to verified that this operations are well defined.

Using Theorem 1.5, [7] we have that $(R_{R^{n-1}},()_+)$ is a commutative $n-$group with querelement of $\frac{a}{s_2^n}$,

$$\overline{\left(\frac{a}{s_2^n}\right)} = \frac{(\dots((a,s_2^n)_{\circ},s_2^n)_{\circ},\dots,s_2^n)_{\circ}}{\overset{(n-1)}{a}}.$$

The multiplication in $R_{R^{n-1}}$ is associative and distributive we respect to n-ary addition $()_+$, therefore $(R_{R^{n-1}},()_+,*)$ is a commutative $(n,2)$-ring.

Let us define $\alpha : R \to R_{R^{n-1}}$, $\alpha(a) = \frac{(a,\overset{(n-1)}{s})_{\circ}}{\overset{(n-1)}{s}}, (\forall) s \in R$. The mapping is correct definite and it is an homomorphism of the $(n,2)-$semirings. Indeed

$$(\alpha(a_1),\dots,\alpha(a_n))_+ = \left(\frac{(a_1,\overset{(n-1)}{s})_{\circ}}{\overset{(n-1)}{s}},\dots,\frac{(a_n,\overset{(n-1)}{s})_{\circ}}{\overset{(n-1)}{s}}\right)_+$$

$$\frac{((a_1,\overset{(n-1)}{s})_{\circ},\dots,(a_n,\overset{(n-1)}{s})_{\circ})_{\circ}}{\overset{(n-1)}{s^{[1]}}}$$

$$= \frac{((a_1^n)_{\circ},\overset{(n-1)}{s^{[1]}})_{\circ}}{\overset{(n-1)}{s^{[1]}}} = \frac{((a_1^n)_{\circ},\overset{(n-1)}{s})_{\circ}}{\overset{(n-1)}{s}} = \alpha((a_1^n)_{\circ})$$

and $\alpha(a \cdot b) = \alpha(a) * \alpha(b)$ because

$$\alpha(a) * \alpha(b) = \frac{(a, \overset{(n-1)}{s})_\circ}{\overset{(n-1)}{s}} * \frac{(b, \overset{(n-1)}{s})_\circ}{\overset{(n-1)}{s}}$$

$$= \frac{(((a, \overset{(n-1)}{s})_\circ \cdot (b, \overset{(n-1)}{s})_\circ, \overset{(n-1)}{s \cdot s})_\circ, \ldots)_\circ, \overset{(n-1)}{s \cdot s})_\circ}{((a, \overset{(n-1)}{s})_\circ \cdot s, \ldots, (a, \overset{(n-1)}{s})_\circ \cdot s, (b, \overset{(n-1)}{s})_\circ \cdot s)_\circ, \ldots, (b, \overset{(n-1)}{s})_\circ \cdot s)_\circ}$$

$$= \frac{(((a, \overset{(n-1)}{s})_\circ \cdot (b, \overset{(n-1)}{s})_\circ, \overset{(n-1)}{s \cdot s})_\circ, \ldots)_\circ, \overset{(n-1)}{s \cdot s})_\circ}{\overset{(n-1)}{((a, \overset{(n-1)}{s})_\circ \cdot s}, \overset{(n-2)}{(b, \overset{(n-1)}{s})_\circ \cdot s)_\circ, (b, \overset{(n-1)}{s})_\circ \cdot s}}$$

$$= \frac{(a \cdot b, \overset{(n-1)}{a \cdot s}, \overset{(n-1)}{b \cdot s}, \overset{(n-1)^2}{s \cdot s}, \overset{(n-1)^2}{s \cdot s})_\circ}{\overset{(n-1)}{((a \cdot s, \overset{(n-1)}{s \cdot s})_\circ, b \cdot s)_\circ}, \overset{(n-1)}{s \cdot s})_\circ, \overset{(n-1)}{b \cdot s}, \overset{(n-1)(n-2)}{s \cdot s}}$$

$$= \frac{((a \cdot b, \overset{(n-1)}{s \cdot s})_\circ, \overset{(n-1)}{a \cdot s}, \overset{(n-1)}{b \cdot s}, \overset{(n-1)(n-2)}{s \cdot s})_\circ}{\overset{(n-1)}{s \cdot s}, \overset{(n-1)}{a \cdot s})_\circ, \overset{(n-1)}{b \cdot s}, \overset{(n-1)(n-2)}{s \cdot s}}$$

$$= \frac{((a \cdot b,), \overset{(n-1)}{s \cdot s},)_\circ}{\overset{(n-1)}{s \cdot s}} = \frac{((a \cdot b,), \overset{(n-1)}{s},)_\circ}{\overset{(n-1)}{s}} = \alpha(a \cdot b)$$

by using associativity , commutativity of operation $()_\circ$ and distributivity operation "$\cdot$" with respect to $()_\circ$.

If $\alpha(a) = \alpha(b)$, therefore $\frac{(a, \overset{(n-1)}{s})_\circ}{\overset{(n-1)}{s}} = \frac{(b, \overset{(n-1)}{s})_\circ}{\overset{(n-1)}{s}}$, that is

$$((a, \overset{(n-1)}{s})_\circ, \overset{(n-1)}{s})_\circ = ((b, \overset{(n-1)}{s})_\circ, \overset{(n-1)}{s})_\circ,$$

by using cancellative properties we have $a = b$, hence α is an injective $(n,2)-$semiring homomorphism. $\square$

Is has the following universal property which determines the $(n,2)-$ring $(R_{R^{n-1}}, ()_+, *)$ up to isomorphism:

Theorem 2. *Let $(R, ()_\circ, \cdot)$ be a commutative $(n,2)-$semiring. If $R_{R^{n-1}}$ is the $(n,2)-$ring above construited , and $\alpha : R \to R_{R^{n-1}}$ is connonical homomorphism defined in Theorem 1, then for any $(n,2)-$ring homomorphism $\beta : R \to R'$, where $(R', [], \circ)$ is a commutative $(n,2)-$ring, there exists an unique ring- homomorphism $\gamma : R_{R^{n-1}} \to R'$, such that $\gamma \circ \alpha = \beta$.*

We remark that the commutativity of n-ary additive operation may be replaced by a weaker condition, namely semicommutativity.

We can state now the sequel result:

Theorem 3. *Any $(n,2)$-ring can be embedded in an $(n,2)$-ring of matrices.*

Proof. Let $(R), (\,)_{+}, \cdot)$ be an $(n,2)-$ ring and $(\mathcal{M}_m(R), [\,]_{+}, *)$ is an $(n,2)-$ ring from Example 3. The mapping $f : R \rightarrow \mathcal{M}_m(R)$

$$f(a) = \begin{pmatrix} a & a & \cdots & a \\ \cdot & \cdot & \cdots & \cdot \\ a & a & \cdots & a \\ \overline{a} & \overline{a} & \cdots & \overline{a} \end{pmatrix}$$

is one - one correspondence between the elements of R and those of $\mathcal{M}_m(R)$.

Moreover, because $(a_1^n)_{\circ} = (\overline{a_1}, \ldots, \overline{a_n})_{\circ}$ we have $f\left((a_1^n)_{\circ}\right) = [f(a_1), \ldots, f(a_n)]_{+}$ and by distributivity of "$\cdot$" with respect to $(\,)_{\circ}$ and by the properties of the querelement in an $(n,2)-$ ring we have

$$\begin{aligned} (\overset{(n-1)}{a \cdot b}, a \cdot \overline{b})_{\circ} &= a \cdot (\overset{(n-1)}{b}, \overline{b})_{\circ} = a \cdot b \\ (\overset{(n-1)}{\overline{a} \cdot b}, \overline{a} \cdot \overline{b})_{\circ} &= \overline{a} \cdot (\overset{(n-1)}{b}, \overline{b})_{\circ} = \overline{a} \cdot b = \overline{a \cdot b}. \end{aligned}$$

Therefore we have

$$\begin{aligned} f(a) * f(b) &= \begin{pmatrix} a & a & \cdots & a \\ \cdot & \cdot & & \cdot \\ a & a & \cdots & a \\ \overline{a} & \overline{a} & \cdots & \overline{a} \end{pmatrix} * \begin{pmatrix} a & a & & a \\ \cdot & \cdot & & \cdot \\ a & a & \cdots & a \\ \overline{a} & \overline{a} & & \overline{a} \end{pmatrix} \\ &= \begin{pmatrix} (\overset{(m-1)}{a \cdot b}, a \cdot \overline{b})_{\circ} & \cdots & (\overset{(m-1)}{a \cdot b}, a \cdot \overline{b})_{\circ} \\ \cdot & \cdots & \cdot \\ (\overset{(m-1)}{a \cdot b}, a \cdot \overline{b})_{\circ} & \cdots & (\overset{(m-1)}{a \cdot b}, a \cdot \overline{b})_{\circ} \\ (\overset{(m-1)}{\overline{a} \cdot b}, \overline{a} \cdot \overline{b})_{\circ} & \cdots & (\overset{(m-1)}{\overline{a} \cdot b}, \overline{a} \cdot \overline{b})_{\circ} \end{pmatrix} \\ &= \begin{pmatrix} a \cdot b & a \cdot b & \cdots & a \cdot b \\ \cdot & \cdot & \cdots & \cdot \\ a \cdot b & a \cdot b & \cdots & a \cdot b \\ \overline{a \cdot b} & \overline{a \cdot b} & \cdots & \overline{a \cdot b} \end{pmatrix} = f(a \cdot b). \end{aligned}$$

The results can be extended to $(n,m)-$rings as well as to generalized $(n,2)$ rings where the addition is only semicommutative.

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