Bul. Ştiinţ. Univ. Baia Mare, Ser. B,
Matematică-Informatică, Vol. XVIII(2002), Nr. 2, 317 - 318

A CHARACTERISTIC PROPERTY OF THE BOOLEAN RING $(\langle M\rangle, \Delta, \cap)$

Vasile POP

Abstract. In [2] we defined the subordination relation on the set of binary operations which are defined on the family of all subsets of a given set. In this paper we show that the only unitary ring structure $(\langle M\rangle, f, g)$ with the property that f is subordinated to the union and g is subordinated to the intersection is the structure of boolean ring.

MSC2000: 03G05,06E20

Keywords: boolean ring, subordination relation

1 Introduction

Let M be an arbitrary set and $\langle M\rangle = \{A \mid A \subseteq M\}$, the family of the subsets of M. On the set $(\langle M\rangle)$ of all binary operations on $\langle M\rangle$ is defined the subordination relation:

If $f, g \in (\langle M\rangle)$ are binary operation on $\langle M\rangle$, we say that *f is subordinated to g or that g subordinated f*, if $f(X,Y) \subset g(X,Y)$ for all $X, Y \in \langle M\rangle$. We denote this by $f \leq g$.

Our purpose is to determine those structures of unitary ring $(\langle M\rangle, f, g)$ on $\langle M\rangle$ with the property $f \leq \cup$ and $g \leq \cap$ (where "$\cup$" is the union and "$\cap$" is the intersection).

2 Main results

An important algebraic structure on $\langle M\rangle$ is the boolean ring $(\langle M\rangle, \Delta, \cap)$, where Δ is the symmetric difference. Using the results obtained in [ ] we can give the following characterization of this ring.

Theorem 1. *If M is a finite set, then the only structure of unitary ring $(\langle M\rangle, f, g)$ on $\langle M\rangle$ with the properties $f \leq \cup$ and $g \leq \cap$ is the boolean ring $(\langle M\rangle, \Delta, \cap)$.*

Proof. Since $(\langle M\rangle, f)$ is a group and the operation f is subordinated to $\cup$ $(f \leq \cup)$, from [2], Theorem 1, it follows that the operation f is unique determined $f(X,Y) = X\Delta Y = (X \cup Y) \setminus (X \cap Y) = (X \setminus Y) \cup (Y \setminus X)$, for all $X, Y \in \langle M\rangle$.

Let us denote $g(X,Y) = X \circ Y$ and E the unit element of the monoid $(\langle M\rangle, \circ)$. We have $X \circ E = X$ and $X \circ E \subset X \cap E$ for all $X \subset M$. So $E = M$ is the unit element of this ring.

If $x, y \in M$, $x \neq y$, then $\{x\} \circ \{y\} \subset \{x\} \cap \{y\} = \emptyset$. Thus $\{x\} \circ \{y\} = \emptyset$.

We have $\{x\} \circ \{x\} \subset \{x\} \cap \{x\} = \{x\}$, whence $\{x\} \circ \{x\} \subset \{\emptyset, \{x\}\}$.

But $\{x\} \circ M = \{x\}$, $M = \{x, y_1, \ldots, y_k\} = \{x\}\Delta\{y_1\}\Delta\ldots\Delta\{y_k\}$. Using the distributivity we have

$$\{x\} \circ M = (\{x\} \circ \{x\})\Delta(\{x\} \circ \{y_1\})\Delta\ldots\Delta(\{x\} \circ \{y_k\}) =$$

$$= (\{x\} \circ \{x\})\Delta\emptyset\Delta\ldots\Delta\emptyset = \{x\} \circ \{x\}$$

Thus $\{x\} \circ \{x\} = \{x\}$.

If $A = \{a_1, \ldots, a_p\}$, $B = \{b_1, \ldots, b_q\}$ using the distributivity we have

$$A \circ B = (\{a_1\}\Delta\ldots\Delta\{a_p\}) \circ (\{b_1\}\Delta\ldots\Delta\{b_q\}) = A \cap B,$$

so $g = \cap$.

Theorem 2. *If M is a finite set, then the unique structure of unitary ring $((M), u, v)$ on (M) with the properties $\cap \leq u$ and $\cup \leq v$ is given by*

$$u(X,Y) = \overline{X\Delta Y} \text{ and } v(X,Y) = X \cup Y.$$

Proof. We have $\overline{X} \cap \overline{Y} \subset u(X,Y)$ and $\overline{X} \cup \overline{Y} \subset v(X,Y)$. Then,

$$\overline{u(X,Y)} \subset X \cup Y, \quad \overline{v(X,Y)} \subset X \cap Y.$$

The operations $\overline{u(\overline{X},\overline{y})}$ and $\overline{v(\overline{X},\overline{y})}$ determines on (M) a ring structure, isomorphic with the ring $((M), u, v)$, induced by the bijective function

$$c: (M) \to (M), \quad c(X) = \overline{X} = M \setminus X.$$

The operations of this ring verify the hypotheses of Theorem 3, so $\overline{u(\overline{X},\overline{y})} = X\Delta Y$ and $\overline{v(\overline{X},\overline{y})} = X \cap Y$. Then:

$$u(X,Y) = \overline{\overline{X}\Delta\overline{Y}} = \overline{X\Delta Y}$$

$$v(X,Y) = \overline{\overline{X} \cap \overline{Y}} = X \cup Y, \quad X, Y \in (M). \quad \square$$

Remark 2. The proofs of the theorems have essentially used the fact that the set M is finite. It is an open problem whether the results hold for infinite sets.

REFERENCES

[1] D. S. Dummit and R. M. Foote, *Abstract Algebra*, 2nd ed., Prentice Hall, Upper Saddle River, New Jersey, 1999.

[2] V. Pop, *A functional characterization of the symmetric difference operation*, Studia Univ. Cluj-Napoca (submitted).

[3] I. Purdea, Gh. Pic, *Treatise of modern algebra*, I, Ed. Acad., Bucureşti, 1977.

Received: 17.09.2002

Department of Mathematics
Technical University
Str. C. Daicoviciu 15, Cluj-Napoca
Romania
E-mail: vasile.pop@utcluj.ro