Bul. Științ. Univ. Baia Mare, Ser. B., Matematică-Informatică, Vol. XVIII(2002), Nr. 2, 319 - 324 perm sell nousible m sellence o 11

A CLASSIFICATION METHOD BASED ON

FUZZY CONTEXTS

If  $y = (Y_1, Y_2, y_3)$  are there similarize **RADELECZKI** limits set X. Let  $\prod_i y_i Y_i, Y_i Y_i$ 

Abstract. The main idea of different fuzzy methods used for the classification of the elements of a finite set A is to define a fuzzy similarity relation among the elements of the set A. In this paper we present a new method for the construction of this similarity relation using some fundamental notions of Fuzzy Concept Analysis.

MSC: 04A72, 06B23

Keywords: similarity relation, partition tree, concept lattice, fuzzy context.

#### 1. Preliminaries

The purpose of this paper is to classify a finite set of objects  $A = \{x_1, x_2, ..., x_n\}$  on the basis of their properties  $P_1, P_2, ..., P_n$   $(n, m \in N)$ . In fact, a classification of the elements means a partition  $\prod = \{A, |1 \le i \le k\}$  of the set A  $(k \le n)$ , where the blocks A, of  $\prod$  are constituted from objects with "similar" properties.

### A) Elements of the theory of fuzzy relations

A binary fuzzy relation  $\rho$  defined between the elements of the sets X and Y is a triple  $\rho = (X, Y, \mu_{\rho})$ , where  $\mu_{\rho} : X \times Y \to [0; 1]$  is a function. The value  $\mu_{\rho}(x, y)$  express the "strength" of the relation  $\rho$  between the elements  $x \in X$  and  $y \in Y$ .

The fuzzy relation  $\rho = (X, Y, \mu_{\rho})$  is said to be *smaller* than the fuzzy relation  $R = (X, Y, \mu_{\rho})$  if  $\mu_{\rho}(x, y) \le \mu_{\chi}(x, y)$  holds for all  $(x, y) \in X \times Y$  is appearable of the smaller of the fuzzy relation  $R = (X, Y, \mu_{\rho})$  if

If X-Y, then  $\rho$  is called homogenous. A fuzzy tolerance (see e.g. [1] or [3]) is a homogenous fuzzy relation  $\rho = (X, X, \mu_{\rho})$  satisfying the properties:

$$\mu_{s,t}(x,x) = \sup \{\min\{\mu_{s,t}(x,y), \mu_{s,t}(y,z)\}, \forall t \in \mathbb{N}, \text{ for each } (x,z) \in \mathbb{N} = \mathbb{N} \}$$

 $\mu_s(x,y) = \mu_s(y,x)$ , for all  $x,y \in X \cap$  multiple years a to usual thin(2) if

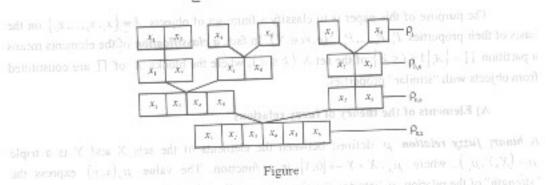
If 
$$\rho$$
 satisfies in addition the inequality  $0.15 \pm 0.00000111VX$  loV scattermobal-scattermobal

then it is called a fuzzy similarity relation (i.e. a fuzzy equivalence - see e. g. [1] or [8]). Let  $\alpha \in [0,1]$ . An  $\alpha$ -cut of a fuzzy relation  $\rho = (X,Y,\mu_o)$  is a crisp (or traditional) binary relation  $\rho_a \subseteq X \times Y$  defined as

$$\rho_{\sigma} = \{(x, y) \in X \times Y \mid \mu_{\sigma}(x, y) \ge \alpha \}. \tag{4}$$

If  $\rho = (X, X, \mu_p)$  is a fuzzy similarity relation, then  $\rho_p$  is an equivalence on the set X. Let  $\prod_a$ stand for the partition induced by  $\rho_a$  on X. It is easy to see that for any  $\alpha' \in [0;1]$  with  $\alpha' \ge \alpha$ ,  $\prod_{s'}$  is a refinement of  $\prod_{s}$ . Therefore to any sequence  $0 \le \alpha_1 < \alpha_2 < ... < \alpha_k \le 1$  we can attach a nested sequence of partitions  $\prod_{x_i}, \prod_{x_i}, \dots, \prod_{x_i}$  and this may be represented in the form of a partition tree, as shown in Figure (the example is from [8]) [2000 and not bodyen won a taxway

$$\mu_{\rho} = \begin{bmatrix} 1 & 0.2 & 1 & 0.6 & 0.2 & 0.6 \\ 0.2 & 1 & 0.2 & 0.2 & 0.8 & 0.2 \\ 10110 & 0.2 & 0.6 & 0.6 & 0.2 & 0.6 \\ 0.6 & 0.2 & 0.6 & 1 & 0.2 & 0.8 \\ 0.2 & 0.8 & 0.2 & 0.2 & 1 & 0.2 \\ 0.6 & 0.2 & 0.6 & 0.8 & 0.2 & 1 \end{bmatrix} \quad \begin{array}{c} \text{ESRob STAM : DSM.} \\ \text{Espansion : Sprowyell.} \\ \text{Espansion : Sprowye$$



strength" of the relation poberwise, the elements of K and vic F The transitive closure of a homogenous fuzzy relation  $\rho = (X, X, \mu_{\rho})$  is the smallest fuzzy relation  $\hat{\rho} = (X, X, \mu_{\rho})$  satisfying the inequality (3) and  $\rho \subseteq \hat{\rho}$ . If  $\rho$  is a fuzzy tolerance, then  $\hat{\rho}$  always exists and it is a fuzzy equivalence. The composition  $\rho \circ \theta$  of two fuzzy relations  $\rho = (X, Y, \mu_{\rho})$  and  $\theta = (Y, Z, \mu_{\rho})$  is defined as a fuzzy relation  $\rho \circ \theta = (X, Z, \mu_{\rho \circ})$ , where:

 $\mu_{orb}(x,z) = \sup \left\{ \min \left\{ \mu_{r}(x,y); \mu_{s}(y,z) \right\} \middle| y \in Y \right\}, \text{ for each } (x,z) \in X \times Z.$ The m-th power of a fuzzy relation  $\rho = (X, X, \mu_p)$  is defined as  $\rho^* = \rho \circ \rho^{-1}$ , m > 1 and  $\rho^i = \rho$ . New let X be a finite set with |X| = n. It is easy to see that there exists a number  $1 \le k \le n$  such that  $\rho^k = \hat{\rho}$ . In this case we also obtain  $\rho^k = \rho^{k+n}$  for all  $m \in N$ .

# B) The principal steps of the fuzzy methods, he may satt much the kell of sense if

The main steps of the several fuzzy classification methods (see e. g. [7]) can be summarised as follows:

1.Let  $A = \{x_1, x_2, ..., x_n\}$  be a finite set of objects. The properties  $P_i$  ( $1 \le i \le m$ ) of the elements of A are defined as fuzzy sets on the universe A characterised by the membership functions  $\mu_A : A \to [0,1]$ ,  $1 \le i \le m$ . The value  $\mu_A(x_i)$  express show much the property  $P_i$  is valid for the object  $x_i \in A$ . Now to any object  $x_i \in A$  is associated a point  $Q_i \in R^n$  defined as  $Q_i = \{\mu_A(x_i), ..., \mu_A(x_i)\}$ 

2. Introducing a metric  $d: R^* \times R^* \to [0,1]$  (this is possible in several ways) a fuzzy tolerance  $\rho = (A, A, \mu_{\rho})$  is defined as follows:

Example 2 and the property of 
$$(x_1, x_2) = T = d(x_1, x_2)$$
 for  $(x_1, x_2) = x_1$  for  $(x_1, x_2) = x_2$  for  $($ 

- 3. Computing the consecutive powers  $\rho^*, \rho^*, ..., \rho^* = (k < n)$  until  $\rho^* = \rho^{kn}$  by using formula (5), the transitive closure  $\hat{\rho}$  of  $\rho$  is obtained as  $\hat{\rho} = \rho^k$ .
- 4. By  $\alpha$ -cuts of this  $\hat{\rho}$ , we produce a sequence of nested partitions  $\prod_{\alpha_1},...,\prod_{\alpha_r}$ , i.e. a partition tree corresponding to a previously established sequence  $0 < \alpha_1 < \alpha_2 < ... < \alpha_r \le 1$ .

## 2. Notions of Formal Concept Analysis

A) Crisp contexts and concept lattices

Given a set G of objects and a set M of attributes (or properties) a binary relation  $I \subset G \times M$  is defined as follows: and the set of the set

 $(g,m) \in I$  if and only if the object  $g \in G$  has the attribute  $m \in M$  . (8)

The triple (G, M, I) is called a *formal context* in mathematical literature (see e.g. [6] or [2]). By defining

$$A' = \left\{ m \in M \mid (g, m) \in I \text{ for all } g \in A \right\}$$

$$B' = \left\{ g \in G \mid (g, m) \in I \text{ for all } m \in B \right\}$$

for all subsets  $A \subseteq G$  and  $B \subseteq M$ , we establish a Galois connection between G an M. The pairs (A, B) with A' = B and B' = A are called the **formal concepts** of the context (G, M, I). The formal concepts of (G, M, I) together with the partial order defined by

see) some settle of 
$$(A_i, B_i) \leq (A_2, B_3) \Leftrightarrow A_i \subseteq A_2$$
 (or equivalently  $B_2 \subseteq B_1$ ) over the given level  $(9)$ 

form a complete lattice L(G, M, I) which is called the concept lattice of the context  $K = (G, M, I)^{|\mathcal{N}_{I}| + |m|^{-2\alpha} \log m^{\alpha}} e_{\lambda} = \frac{1}{2\alpha}, \text{ matrix of the new and finite of } e_{\lambda} = \frac{1}{2\alpha}, \text{ and then } e_{\lambda} \ge 3 \ge 1$ 

Remark: If  $A = A^*$ , then the pair (A, A') is a formal concept of the context (G, M, I). For any  $g \in G$  we define the concept  $\gamma(g) = (g)^n, (g)$ . It is easy to see that  $\gamma(g)$  is the smallest concept (A, B) with  $g \in A$ .

elements of a sire detect as these are on the gang over A characterised by the members B) Fuzzy contexts and concept lattices

The general formulation of the notions below can be found in [4]. According to our aim here we present them only in a particular form: " I hold one of work has a toulde of and hills A fuzzy context is a triple (G, M, I) where G is a set of objects, M is a set of attributes and

 $I = (G, M, \mu_i)$  is a binary fuzzy relation defined by a membership function  $\mu_i : G \times M \rightarrow [0;1]$ . The value  $\mu_i(g,m)$  express "how much is valid" the attribute  $m \in M$  for the object  $g \in G$ . For each  $\alpha \in [0;1]$  the  $\alpha$ -cut  $I_{\alpha} = \{(g,m) | \mu_i(g,m) \ge \alpha\}$  determines a "traditional" context  $K_{\sigma} = (G, M, I_{\sigma})$  and a "traditional" or crisp concept lattice  $L_{\sigma} = (G, M, I_{\sigma})$  (corresponding to

the context  $K_a$ ). In our particular case the fuzzy concept lattice  $\mathcal{L}(G,M,I)$  of the fuzzy context (G,M,I) is defined by identifying it to the set  $\{((G, M, I), \alpha) \mid \alpha \in [0, 1]\}$  corresponding to all concept lattices of the fuzzy context K = (G, M, I) [5]. (For a more detailed formulation see [4].)

# 3. The principle of our classification method

Given a finite set  $A = \{x_1, x_2, \dots, x_n\}$  of objects and a finite set  $M = \{P_1, P_2, \dots, P_n\}$  of attributes interpreted as fuzzy sets with universe A and with different membership functions  $\mu_e: A \rightarrow [0;1]$ ,  $1 \le i \le m$ , a fuzzy relation  $I = (A, M, \mu_i)$  and a fuzzy context K = (A, M, I)is defined as follows: notation of the property of the proper

$$\mu_{i}(x_{i}, P_{i}) = \mu_{p}(x_{i})$$
 (10)

Let  $\gamma_a(x_i)$  associate the concept  $(\{x_i\}^n, \{x_i\}^n)$  defined by the crisp context  $K_a = (G, M, I_a)$ , where  $\alpha \in [0;1]$ .

Further, we consider a fuzzy set  $M(x_i)$  with universe M to any object  $x_i \in A$ , by defining its membership function() = M = M [0;1] aso) the bells a sure a - L bas & - L thus (& . ) susa

$$\mu_{j_i}(P_i) = \mu_i(x_i, P_i)$$
, for all  $P_i \in M$ ,  $(1 \le k \le m \land h)$  to eigensteen sum (11) (1)

The similarity of two fuzzy sets  $M(x_j)$  and  $M(x_j)$  is defined as it is usual in literature (see

e.g. [1]): 
$$28(1, 20) \operatorname{constant} \operatorname{End}(1, 10) \operatorname{End}(1$$

We note that  $S(A, B) = 1 - ||A\nabla B|| = 1$  iff A = B.

Now we define a fuzzy tolerance  $T = (A, A, \mu_{\tau})$  as follows

$$\mu_{c}(x_{c}, x_{c}) := S(M(x_{c}), M(x_{c})) \sup \{\alpha \in [0;1] | \gamma_{a}(x_{c}) = \gamma_{c}(x_{c})\}$$
(12)

Clearly, the above supremum always exits, and we have  $\mu_T(x_i, x_j) = \mu_T(x_j, x_j) \in [0; 1]$ by definition. Since  $S(M(x_i), M(x_i)) = 1$  and since  $\gamma_{\alpha}(x_i) = \gamma_{\alpha}(x_i)$  holds for all  $\alpha \in [0; 1]$ , we get  $\mu_r(x_i, x_i) = 1$ , for all  $x_i \in A$  - proving that T is a fuzzy tolerance.

In what follows, our construction uses the same steps as the formerly presented fuzzy methods (see Subsection 1.B), for instance, we proceed constructing a fuzzy similarity relation  $S = (A, M, \mu_e)$  by computing the powers  $T^1, T^1, ..., T^k$  of the fuzzy tolerance T until  $T^i = T^{i+1}$ .

Concluding remarks: The origin of our method comes from an application of the fuzzy contexts in Group Technology, namely, to classify some technological objects on the basis of their common attributes [5].

The advantage of the method consists in the fact that it does not need the construction of an additional R" metric used by the majority of fuzzy methods.

Acknowledgement: The support by Hungarian National Foundation for Scientific Research (Grant No. T029525, T030243 and T034137) and by István Széchenyi Grant of Hungarian Academy of Science is gratefully acknowledged. The author wishes to express his thanks to professor T. Toth for his advice.

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Received: 26.10.2002

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readly, the above squeezidin shows such and so have  $\varphi_1(x, x_1) = \varphi_1(x, x_2) = [0, 1]$  is definition where  $\varphi(x) = \varphi(x) = \varphi(x) = \varphi(x)$  holds for all  $\alpha \in [0, 1]$ , we have  $\varphi(x) = \varphi(x) = \varphi(x) = \varphi(x)$  independently  $\varphi(x) = \varphi(x) = \varphi(x)$  independently.

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