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A CLASSIFICATION METHOD BASED ON FUZZY CONTEXTS

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Abstract. The main idea of different fuzzy methods used for the classification of the elements of a finite set A is to define a fuzzy similarity relation among the elements of the set A. In this paper we present a new method for the construction of this similarity relation using some fundamental notions of Fuzzy Concept Analysis.

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1. Preliminaries

The purpose of this paper is to classify a finite set of objects $A = \{x_1, x_2, ..., x_n\}$ on the basis of their properties $P_1, P_2, ..., P_m$ $(n, m \in N)$. In fact, ***a classification*** of the elements means a partition $\Pi = \{A_i | 1 \le i \le k\}$ of the set A $(k \le n)$, where the blocks A_i of Π are constituted from objects with "similar" properties.

A) Elements of the theory of fuzzy relations

A ***binary fuzzy relation*** ρ defined between the elements of the sets X and Y is a triple $\rho = (X, Y, \mu_\rho)$, where $\mu_\rho : X \times Y \to [0;1]$ is a function. The value $\mu_\rho(x, y)$ express the "strength" of the relation ρ between the elements $x \in X$ and $y \in Y$.

The fuzzy relation $\rho = (X, Y, \mu_\rho)$ is said to be ***smaller*** than the fuzzy relation $R = (X, Y, \mu_R)$ if $\mu_\rho(x, y) \le \mu_R(x, y)$ holds for all $(x, y) \in X \times Y$.

If X=Y, then ρ is called homogenous. A ***fuzzy tolerance*** (see e.g. [1] or [3]) is a homogenous fuzzy relation $\rho = (X, X, \mu_\rho)$ satisfying the properties:

$$\mu_\rho(x, x) = 1 \text{, for all } x \in X \qquad (1)$$

and

$$\mu_\rho(x, y) = \mu_\rho(y, x) \text{, for all } x, y \in X \qquad (2)$$

If ρ satisfies in addition the inequality

$$\mu_\rho(x,z) \geq \min\{\mu_\rho(x,y), \mu_\rho(y,z)\}, \text{ for all } x,y,z \in X \tag{3}$$

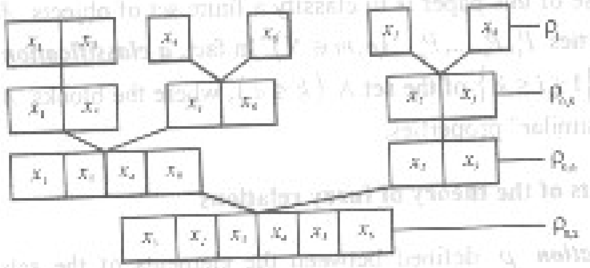
then it is called a ***fuzzy similarity relation*** (i.e. a ***fuzzy equivalence*** - see e. g. [1] or [8]).

Let $\alpha \in [0;1]$. An ***α-cut*** of a fuzzy relation $\rho = (X, Y, \mu_\rho)$ is a ***crisp*** (or traditional) binary relation $\rho_\alpha \subseteq X \times Y$ defined as

$$\rho_\alpha = \{(x,y) \in X \times Y \mid \mu_\rho(x,y) \geq \alpha\}. \tag{4}$$

If $\rho = (X, X, \mu_\rho)$ is a fuzzy similarity relation, then ρ_α is an equivalence on the set X. Let $\prod_\alpha$ stand for the partition induced by ρ_α on X. It is easy to see that for any $\alpha' \in [0;1]$ with $\alpha' \geq \alpha$, $\prod_{\alpha'}$ is a refinement of $\prod_\alpha$. Therefore to any sequence $0 \leq \alpha_1 < \alpha_2 < \ldots < \alpha_k \leq 1$ we can attach a nested sequence of partitions $\prod_{\alpha_1}, \prod_{\alpha_2}, \ldots, \prod_{\alpha_k}$ and this may be represented in the form of a ***partition tree***, as shown in Figure (the example is from [8]).

$$\mu_\rho = \begin{bmatrix} 1 & 0.2 & 1 & 0.6 & 0.2 & 0.6 \\ 0.2 & 1 & 0.2 & 0.2 & 0.8 & 0.2 \\ 1 & 0.2 & 1 & 0.6 & 0.2 & 0.6 \\ 0.6 & 0.2 & 0.6 & 1 & 0.2 & 0.8 \\ 0.2 & 0.8 & 0.2 & 0.2 & 1 & 0.2 \\ 0.6 & 0.2 & 0.6 & 0.8 & 0.2 & 1 \end{bmatrix}$$



Figure

The ***transitive closure*** of a homogenous fuzzy relation $\rho = (X, X, \mu_\rho)$ is the smallest fuzzy relation $\hat{\rho} = (X, X, \mu_{\hat{\rho}})$ satisfying the inequality (3) and $\rho \subseteq \hat{\rho}$. If ρ is a fuzzy tolerance, then $\hat{\rho}$ always exists and it is a fuzzy equivalence. The ***composition*** $\rho \circ \theta$ of two fuzzy relations $\rho = (X, Y, \mu_\rho)$ and $\theta = (Y, Z, \mu_\theta)$ is defined as a fuzzy relation $\rho \circ \theta = (X, Z, \mu_{\rho \circ \theta})$, where:

$$\mu_{\rho\circ\theta}(x,z) = \sup\{\min\{\mu_\rho(x,y); \mu_\theta(y,z)\} \mid y \in Y\}, \text{ for each } (x,z) \in X \times Z. \tag{5}$$

The ***m-th power*** of a fuzzy relation $\rho = (X, X, \mu_\rho)$ is defined as $\rho^m = \rho \circ \rho^{m-1}$, $m > 1$ and

$\rho^1 = \rho$. Now let X be a finite set with $|X| = n$. It is easy to see that there exists a number $1 \le k \le n$ such that $\rho^k = \hat{\rho}$. In this case we also obtain $\rho^k = \rho^{k+m}$ for all $m \in N$.

B) The principal steps of the fuzzy methods

The main steps of the several fuzzy classification methods (see e. g. [7]) can be summarised as follows:

1.Let $A = \{x_1, x_2, \ldots, x_n\}$ be a finite set of objects. The properties $P_i \ (1 \le i \le m)$ of the elements of A are defined as fuzzy sets on the universe A characterised by the membership functions $\mu_{P_i} : A \to [0;1]$, $1 \le i \le m$. The value $\mu_{P_i}(x_j)$ express "how much" the property P_i is valid for the object $x_j \in A$. Now to any object $x_j \in A$ is associated a point $Q_j \in R^m$ defined as

$$Q_j = (\mu_{P_1}(x_j), \ldots, \mu_{P_m}(x_j)) \quad (6)$$

2. Introducing a metric $d : R^m \times R^m \to [0;1]$ (this is possible in several ways) a fuzzy tolerance $\rho = (A, A, \mu_\rho)$ is defined as follows:

$$\mu_\rho(x_i, x_j) = 1 - d(x_i, x_j) \quad (7)$$

3. Computing the consecutive powers $\rho^2, \rho^3, \ldots, \rho^k \ (k < n)$ until $\rho^k = \rho^{k+1}$ by using formula (5), the transitive closure $\hat{\rho}$ of ρ is obtained as $\hat{\rho} = \rho^k$.

4. By α-cuts of this $\hat{\rho}$, we produce a sequence of nested partitions $\Pi_{\alpha_1}, \ldots, \Pi_{\alpha_s}$, i.e. a partition tree corresponding to a previously established sequence $0 < \alpha_1 < \alpha_2 < \ldots < \alpha_s \le 1$.

2. Notions of Formal Concept Analysis

A) Crisp contexts and concept lattices

Given a set G of objects and a set *M* of attributes (or properties) a binary relation $I \subseteq G \times M$ is defined as follows:

$(g, m) \in I$ if and only if the object $g \in G$ has the attribute $m \in M$. (8)

The triple (G, M, I) is called a ***formal context*** in mathematical literature (see e.g. [6] or [2]). By defining

$$A' = \{m \in M \mid (g, m) \in I \text{ for all } g \in A\}$$
$$B' = \{g \in G \mid (g, m) \in I \text{ for all } m \in B\}$$

for all subsets $A \subseteq G$ and $B \subseteq M$, we establish a Galois connection between *G* an *M*. The pairs (*A*, *B*) with $A' = B$ and $B' = A$ are called the ***formal concepts*** of the context (G, M, I). The formal concepts of (G, M, I) together with the partial order defined by

$$(A_1, B_1) \le (A_2, B_2) \Leftrightarrow A_1 \subseteq A_2 \text{ (or equivalently } B_2 \subseteq B_1) \quad (9)$$

form a complete lattice $Ł(G, M, I)$ which is called the ***concept lattice*** of the context $K = (G, M, I)$.

Remark: If $A = A''$, then the pair (A, A') is a formal concept of the context (G, M, I).

For any $g \in G$ we define the concept $\gamma(g) = \left(\{g\}'', \{g\}'\right)$. It is easy to see that $\gamma(g)$ is the smallest concept (A, B) with $g \in A$.

B) Fuzzy contexts and concept lattices

The general formulation of the notions below can be found in [4]. According to our aim here we present them only in a particular form:

A ***fuzzy context*** is a triple (G, M, I) where G is a set of objects, M is a set of attributes and $I = (G, M, \mu_I)$ is a binary fuzzy relation defined by a membership function $\mu_I : G \times M \to [0;1]$. The value $\mu_I(g, m)$ express "how much is valid" the attribute $m \in M$ for the object $g \in G$.

For each $\alpha \in [0;1]$ the α-cut $I_\alpha = \{(g, m) \mid \mu_I(g, m) \geq \alpha\}$ determines a "traditional" context $K_\alpha = (G, M, I_\alpha)$ and a "traditional" or ***crisp*** concept lattice $Ł_\alpha = (G, M, I_\alpha)$ (corresponding to the context K_α).

In our particular case the fuzzy concept lattice $Ł(G, M, I)$ of the fuzzy context (G, M, I) is defined by identifying it to the set $\{Ł((G, M, I), \alpha) \mid \alpha \in [0;1]\}$ corresponding to all concept lattices of the fuzzy context $K = (G, M, I)$ [5]. (For a more detailed formulation see [4].)

3. The principle of our classification method

Given a finite set $A = \{x_1, x_2, \ldots, x_n\}$ of objects and a finite set $M = \{P_1, P_2, \ldots, P_m\}$ of attributes interpreted as fuzzy sets with universe A and with different membership functions $\mu_{P_i} : A \to [0;1]$, $1 \leq i \leq m$, a fuzzy relation $I = (A, M, \mu_I)$ and a fuzzy context $K = (A, M, I)$ is defined as follows:

$$\mu_I(x_i, P_j) = \mu_{P_j}(x_i) \tag{10}$$

Let $\gamma_\alpha(x_i)$ associate the concept $\left(\{x_i\}'', \{x_i\}'\right)$ defined by the crisp context $K_\alpha = (G, M, I_\alpha)$, where $\alpha \in [0;1]$.

Further, we consider a fuzzy set $M(x_i)$ with universe M to any object $x_i \in A$, by defining its membership function $\mu_{x_i} : M \to [0;1]$ as

$$\mu_{x_i}(P_k) = \mu_I(x_i, P_k), \text{ for all } P_k \in M, \ 1 \leq k \leq m. \tag{11}$$

The ***similarity of two fuzzy sets*** $M(x_i)$ and $M(x_j)$ is defined as it is usual in literature (see e.g. [1]):

$$S(M(x_i),M(x_j)) = 1 - \|M(x_i)\nabla M(x_j)\| = 1 - \frac{\sum_{k=1}^{m}\left[\mu_i(P_k) - \mu_j(P_k)\right]}{|M|}$$

$$= 1 - \frac{1}{m}\sum_{k=1}^{m}\left|\mu_i(P_k) - \mu_j(P_k)\right|.$$

We note that $S(A,B) = 1 - \|A\nabla B\| = 1$ iff $A=B$.

Now we define a fuzzy tolerance $T = (A, A, \mu_T)$ as follows

$$\mu_T(x_i,x_j) := S(M(x_i),M(x_j))\cdot \sup\left\{\alpha \in [0;1] \mid \gamma_\alpha(x_i) = \gamma_\alpha(x_j)\right\} \tag{12}$$

Clearly, the above supremum always exits, and we have $\mu_T(x_i,x_j) = \mu_T(x_j,x_i) \in [0;1]$ by definition. Since $S(M(x_i),M(x_i)) = 1$ and since $\gamma_\alpha(x_i) = \gamma_\alpha(x_i)$ holds for all $\alpha \in [0;1]$, we get $\mu_T(x_i,x_i) = 1$, for all $x_i \in A$ - proving that T is a fuzzy tolerance.

In what follows, our construction uses the same steps as the formerly presented fuzzy methods (see Subsection 1.B), for instance, we proceed constructing a fuzzy similarity relation $S = (A, M, \mu_s)$ by computing the powers $T^1, T^2, \ldots, T^k$ of the fuzzy tolerance T until $T^k = T^{k+1}$.

Concluding remarks: The origin of our method comes from an application of the fuzzy contexts in Group Technology, namely, to classify some technological objects on the basis of their common attributes [5].

The advantage of the method consists in the fact that it does not need the construction of an additional R^m metric used by the majority of fuzzy methods.

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REFERENCES

[1] Dubois, D., Prade, H.: *Fuzzy Sets and Systems, Theory and Applications; Mathematics in Science and Engineering*, Vol. 144, Academic Press, New York, 1980.

[2] Ganter, B., Wille R.: *Formal Concept Analysis;* Mathematical Foundations, Springer Verlag, Berlin, 1999.

[3] Negoiţa, C. V. and Ralescu, D. A.: *Applications of Fuzzy Sets to System Analysis*, Birkhauser, Basel (1975).

[4] Pollandt, S.: *Fuzzy-Begriffe*, Formale Begriffsanalyse Unscharfer Daten, Springer Verlag, Berlin, 1997.

[5] Radeleczki, S. and Tóth, T.: *Concept lattices and fuzzy methods and their application in Group Technology*, Research report, Miskolc University, 1999 (in Hungarian).

[6] Wille, R.: *Restructuring lattice theory: an approach based on hierarchies of concepts*, In: I.

Rival (ed.), Ordered Sets, 445-470, Reidel, Dordrecht-Boston, 1982.
[7] **Xu, H.** and **Wang, H. P.:** *Part family formation for GT applications based on fuzzy mathematics*, Int. J. Prod. Res. Vol. 27, No. 9. (1989), 1637-1651.
[8] **Zadeh, L. A.:** *Similarity relations and fuzzy orderings*, Inf. Sci. 3 (1971), 177-200.

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