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OPTIMIZATION PROBLEMS WITH INFINITELY MANY CONSTRAINTS

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Abstract. The paper intends to give a survey on some important aspects of optimization problems with infinitely many constraints. We consider the structure of the problem, optimality conditions, Newton methods based on these conditions and related genericity properties.

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1 Introduction

An optimization problem is a problem of the type:

$$\max \ f(x) \qquad \text{subject to } x \in \mathcal{F}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is the objective function and $\mathcal{F} \subset \mathbb{R}^n$ is a given set, the so-called *feasible set*. To be computationally practicable we assume that the set $\mathcal{F}$ is given explicitely by inequalities (we omit equality constraints for brevity). If $\mathcal{F}$ is defined by finitely many constraints we call P a *finite program* (FP),

$$\text{FP:} \qquad \mathcal{F} = \{x \in \mathbb{R}^n \mid g_j(x) \le 0, \ j \in J\} ,$$

where $J = \{1, \ldots, q\}$ is a finite index set. The problem is said to be *semi-infinite* if $\mathcal{F}$ is given by,

$$\text{SIP:} \qquad \mathcal{F} = \{x \in \mathbb{R}^n \mid g(x, y) \le 0, \ \text{ for all } y \in Y\} ,$$

where $Y \subset \mathbb{R}^m$ is a (possibly) infinite index set. If in addition, the index set $Y = Y(x)$ depends on the variable x, the program is a semi-infinite problem with *variable index set* or *generalized semi-infinite problem* (GSIP) for short,

$$\text{GSIP:} \qquad \mathcal{F} = \{x \in \mathbb{R}^n \mid g(x, y) \le 0, \ \text{ for all } y \in Y(x)\} .$$

Let the index sets be given by

$$Y(x) = \{y \in \mathbb{R}^m \mid v_l(x, y) \le 0, \ l \in L := \{1, \ldots, k\}\} .$$

(Note that in case of SIP the functions v_l do not depend on x.) We assume that all problem functions f, g, v_l are C^∞ functions. For simplicity throughout we assume that there is a compact set $Y_c \subset \mathbb{R}^m$ such that

$$Y(x) \subset Y_c \qquad \text{for all } x \in \mathbb{R}^n.$$

Recall that a feasible point $\overline{x} \in \mathcal{F}$ is said to be a local maximizer if there is a neighborhood U of $\overline{x}$ such that

$$f(x) \leq f(\overline{x}) \quad \text{for all } x \in \mathcal{F} \cap U .$$

Semi-infinite programming is a topic in continuous (non-linear) optimization. Since the 60-th more than 2000 papers appeared on this subject. We refer the reader to the survey [3] containing many references. GSIP is only studied since about 1985, see e.g. [4] , [7], [9].

2 Applications of semi-infinite problems

In this section we present a few examples/applications of semi-infinite programming. For instance, Chebyshev approximation problems lead to semi-infinite problems, but also to GSIP.

Example 1. *(Direct and reverse Chebyshev approximation)*
Consider the approximation problem

AP: *We wish to approximate $h(y) \in C^\infty(\mathbb{R}^m, \mathbb{R})$ on the compact set $Y \subset \mathbb{R}^m$ by approximating functions $a(x,y) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R})$ in the max-norm (Chebyshev-norm),*

$$\min_{x \in \mathbb{R}^n} \max_{y \in Y} \left| h(y) - a(x,y) \right|$$

By introducing the extra variable ϵ (the error), AP takes the form of a SIP:

$$\min_{x,\epsilon} \epsilon \quad s.\ t. \quad g^{\pm}(x,y) := \pm\big(h(y) - a(x,y)\big) \leq \epsilon \quad \text{for all } y \in Y .$$

The so-called reverse Chebyshev problem consists of fixing the approximation error ϵ and making the region Y as large as possible (see [5]). Suppose, the set $Y = Y(d)$ is parameterized by $d \in \mathbb{R}^p$ and $v(d)$ denotes the volume of $Y(d)$ (e.g. $Y(d) = [-d_1, d_1] \times [-d_2, d_2]$). The reverse Chebyshev problem then leads to the GSIP (ϵ fixed):

$$\max_{d,x} v(d) \quad s.\ t. \quad g^{\pm}(x,y) := \pm\big(h(y) - p(x,y)\big) \leq \epsilon \quad \text{for all } y \in Y(d).$$

Many control problems in robotics lead to semi-infinite problems (cf. [2]). The next example studies an application of semi-infinite programming in Partial Differential Equation.

Example 2. *Consider the following Boundary-value problem,*

BVP: *Given $G_0 \subset \mathbb{R}^2$ (G_0 a simply connected region with smooth boundary ∂G_0) and $k > 0$. Find a function $u \in C^2(\overline{G_0})$ such that with the Laplacian $\Delta u = u_{y_1 y_1} + u_{y_2 y_2}$:*

$$\begin{aligned} \Delta u(y) &= k, \quad \forall y \in G_0 \\ u(y) &= 0, \quad \forall y \in \partial G_0 \end{aligned}$$

By choosing a linear space of appropriate trial functions $u_j \in C^2(\mathbb{R}^2)$, $S_n = \{u(x,y) = \sum_{j=1}^{n} x_j u_j(y)\}$, this problem BVP can approximately be solved via the following SIP-program.

$$\text{SIP:} \qquad \min_{\varepsilon, x} \varepsilon \quad \text{s.t.} \quad \begin{aligned} \pm(\Delta u(x,y) - k) &\leq \varepsilon, \quad \forall y \in G_0 \\ \pm u(x,y) &\leq \varepsilon, \quad \forall y \in \partial G_0 \end{aligned}$$

In [1] the related but more complicated so-called Shape Optimization Problem *has been considered theoretically.*

SOP: *Find a (simply connected) region $G \subset \mathbb{R}^2$ with normalized area $\mu(G) = 1$ and a function $u \in C^2(\overline{G})$ which solves with some given function $F(G,u)$ the problem*

$$\min_{\mu(G)=1, u} F(G,u) \quad \text{s.t.} \quad \begin{aligned} \Delta u(y) &= k, \quad \forall y \in G \\ u(y) &= 0, \quad \forall y \in \partial G \end{aligned}$$

This is a problem with variable region G and leads to the following GSIP-problem: Choose some appropriate family of regions $G(\alpha)$ depending on a parameter $\alpha \in \mathbb{R}^p$ and satisfying $\mu(G(\alpha)) = 1$ for all α. Then, fix some small error $\varepsilon > 0$ and solve with the trial functions $u(x,y) \in S_n$:

$$\min_{\alpha, x} F(G(\alpha), u(x,\cdot)) \quad \text{s.t.} \quad \begin{aligned} \pm(\Delta u(x,y) - k) &\leq \varepsilon, \quad \forall y \in G(\alpha) \\ \pm u(x,y) &\leq \varepsilon, \quad \forall y \in \partial G(\alpha) \end{aligned}$$

All the examples can be formulated in a more geometrical setting as follows: Given a family of sets $S(x) \subset \mathbb{R}^m$ depending on $x \in \mathbb{R}^n$, find a 'body' Y of a given form and a value $\overline{x}$ such that Y is contained in $S(\overline{x})$ and Y is as large as possible.

Suppose $S(x)$ is defined by

$$S(x) = \{y \in \mathbb{R}^m \mid g(x,y,t) \leq 0, \quad \text{for all } t \in Q\}$$

where Q is a given compact set in $\mathbb{R}^s$ and $g \in C^2(\mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^s, \mathbb{R})$. Let the body $Y(d) \subset \mathbb{R}^m$ be parameterized by $d \in \mathbb{R}^q$ (by finitely many inequalities). Let $v(d)$ be a measure of the size of $Y(d)$ (e.g. the volume). To maximize $v(d)$ for $Y(d) \subseteq S(x)$ becomes:

$$\max_{d,x} v(d) \quad \text{s.t.} \quad g(x,y,t) \leq 0 \quad \text{for all } y \in Y(d),\ t \in Q$$

3 Structure of the feasible set

This section intends to provide an overview on the structure of the feasible set of semi-infinite problems. We will see that the feasible set in GSIP may have a much more complicated structure than in common SIP.

Lemma 1. *The feasible set $\mathcal{F}$ of a FP or SIP is closed.*

This follows directly by the fact that the feasible set $\mathcal{F}$ of SIP is the intersection of the closed sets $\{x \mid g(x,y) \le 0\}$ (fixed $y \in Y$). Moreover, this implies that in general (generic case) $\mathcal{F}$ cannot possess so-called 're-entrant' corner points.

REMARK. It can be shown, that a set $\mathcal{F}$ in $\mathbb{R}^n$ is closed and convex if and only $\mathcal{F}$ is defined by (possibly infinitely many) linear inequalities, i.e., $\mathcal{F} = \{a^T(y)x \le b(y),\ y \in Y\}$.

For the case of GSIP the properties of the feasible set strongly depends on the analytic behavior of the index set $Y(x)$. It will appear that the following holds:

- If the mapping $Y : \mathbb{R}^n \to 2^{\mathbb{R}^m}$ is not continuous then the feasible set $\mathcal{F}$ need not be closed.

  If Y is continuous but not differentiable then the set $\mathcal{F}$ is closed but may still have the structure of a union of feasible sets in ordinary FP or SIP (structure of disjunctive optimization problems, with re-entrant corner points in $\mathcal{F}$).

  If Y behaves 'differentiable' then P is (theoretically) equivalent with an ordinary SIP.

  Under further regularity conditions (see reduction approach in Section 4) the problem GSIP can locally be described by a finite program.

We begin our structural analysis with a simple example of a GSIP with non-closed feasible set.

Example 1. Consider the problem

$$\min_{x \in \mathbb{R}} x \quad \text{s.t.} \quad x \ge 1 - y, \quad \forall y \in Y(x) = \{y \in \mathbb{R} \mid y \ge 0,\ y \le -x\}$$

Obviously, the feasible set consists of the open interval $(0, \infty)$. A minimizer of the program does not exist. The problem here is that the index set $Y(x)$, is not continuous at $x = 0$.

If $Y(x)$ is continuous (see e.g. [12] for a definition) then non-closedness of $\mathcal{F}$ is excluded. We omit the simple proof.

Lemma 2. *Suppose the mapping $Y : K \to 2^{\mathbb{R}^m}$ is continuous on a compact set $K \subseteq \mathbb{R}^n$. Then the set $\mathcal{F} \cap K$ is compact (in particular closed).*

If $Y(x)$ is continuous but not 'differentiable' the set $\mathcal{F}$ of GSIP may have re-entrant corner points (the structure of a disjunctive problem). We give an illustrative example.

Example 2:

$$\min_{x \in \mathbb{R}^2} x_1^2 + (x_2 - 1)^2 \quad \text{s.t.} \ y - x_2 \ge 0, \quad \forall y \in Y(x)$$

with the index set $Y(x) = \{y \in \mathbb{R} \mid y - x_1 \ge 0,\ y + x_1 \ge 0\}$. The set $Y(x)$ behaves Lipschitz continuous but is not smooth at points $(0, x_2)$. The feasible set is $\mathcal{F} = \{x \in \mathbb{R}^2 \mid |x_1| \ge x_2\}$

with a re-entrant corner point at $x = (0,0)$. The solutions of the problem are the points $\bar{x} = (\pm 1/2, 1/2)$. Obviously the problem is equivalent with the program in disjunctive form (not intersection but union of half spaces):

$$\min \; x_1^2 + (x_2 - 1)^2 \quad \text{s.t.} \; x \in \{x \mid x_2 \le x_1\} \cup \{x \mid x_2 \le -x_1\}.$$

If however the index set $Y(x)$ is defined by e.g.

$$Y(x) = \{y = T(x,y) \mid y \in Y\} \quad \text{with (fixed) compact } Y \subset \mathbb{R}^m$$

and a smooth function $T : \mathbb{R}^n \times Y \to \mathbb{R}^m$ than the feasibility condition becomes

$$\hat{g}(x,y) := g(x, T(x,y)) \le 0, \quad \forall y \in Y$$

and the GSIP problem reduces to SIP.

Under some convexity assumptions, the following holds for the feasible set $\mathcal{F}$ of common SIP (direct from the definition of convexity):

If for each fixed $y \in Y$ the function$g(\cdot, y)$ is convex in x then $\mathcal{F}$ is convex.

Example 2 shows that the situation is more complicated for GSIP. Convexity of $\mathcal{F}$ only follows under very strong conditions on the set mapping $Y(x)$ (we omit the straightforward proof; see also [12]):

Suppose that the function $g(x,y)$ is convex in (x,y) (on $\mathbb{R}^{n+m}$) and the following set-valued inclusion holds: For any $x_1, x_2 \in \mathbb{R}^n$ and α, $0 < \alpha < 1$ we have,

$$Y(\alpha x_1 + (1-\alpha)x_2) \subset \alpha Y(x_1) + (1-\alpha)Y(x_2)\,. \tag{1}$$

Then, the feasible set $\mathcal{F}$ of GSIP is convex.

4 Optimality conditions and Newton method

The so-called Kuhn-Tucker method for solving optimization problems is based on the following principle:

- Apply Newton type iterations to a system of (first order) optimality conditions.

The convergence behavior of this approach strongly depends on the fact whether certain regularity conditions are satisfied at the solution of the program or not. So it is important to know whether in general (genericly) these regularity conditions hold. Let us begin with considering the case of unconstrained optimization:

UP: (unconstrained program) We wish to find a local maximizer: $\max_{x \in \mathbb{R}^n} f(x)$.

A necessary condition for a local maximizer is:

$$\nabla f(x) = 0$$

To solve this system we may use the *Newton method:* $x_{k+1} = x_k - (\nabla^2 f(x_k))^{-1} \nabla f(x_k)$. It is well-known that this process (locally) converges quadratically to a solution $\overline{x}$ of $\nabla f(\overline{x}) = 0$ if:

$$\nabla^2 f(\overline{x}) \text{ is regular.}$$

The question is now whether we can expect this regularity condition to hold in *general* (generic case). To answer this we consider the

Problem set: $\mathcal{P} = \{f \in C^\infty\} \equiv C^\infty(\mathbb{R}^n, \mathbb{R})$ endowed with the so-called strong Withney topology (see [8]).

The following gives a positive answer to our question and forms the theoretical basis for the Newton-type methods. By a generic subset of a set $\mathcal{P}$ we mean a subset which is dense and open in $\mathcal{P}$.

Theorem 1. (Jongen/Jonker/Twilt *see [8]) There is an open and dense subset* $\mathcal{P}_0 \subseteq C^\infty(\mathbb{R}^n, \mathbb{R})$ *such that for all* $f \in \mathcal{P}_0$:

$$\|\nabla f(x)\| + |\det \nabla^2 f(x)| > 0, \forall x \in \mathbb{R}^n$$

Let us now turn to finite programs. To formulate optimality conditions we define the *active index set*, $J_0(\overline{x}) = \{j \in J \mid g_j(\overline{x}) = 0\}$. The following is the F. John necessary optimality condition: If $\overline{x} \in \mathcal{F}$ is a local solution of FP then there are multipliers $\overline{\mu}_0 \geq 0$, $\overline{\mu}_j \geq 0, j \in J_0(\overline{x})$ such that

$$\overline{\mu}_0 \nabla f(\overline{x}) - \sum_{j \in J_0(\overline{x})} \overline{\mu}_j \nabla g_j(\overline{x}) = 0, \quad \overline{\mu}_0 + \sum_{j \in J_0(\overline{x})} \overline{\mu}_j = 1 \tag{2}$$

A regularity condition at a feasible point $\overline{x} \in \mathcal{F}$ is the *linear independence constraint qualification* (LICQ) at $\overline{x}$: $\nabla g_j(\overline{x}), \quad j \in J_0(\overline{x})$ are linearly independent vectors.

Under this constraint qualification LICQ at a local minimizer $\overline{x}$ the F. John condition can be replaced by the KKT-conditions:

$$\begin{aligned} \nabla_x L(\overline{x}, \overline{\mu}) := \nabla f(\overline{x}) - \sum_{j \in J_0(\overline{x})} \overline{\mu}_j \nabla g_j(\overline{x}) &= 0, \\ g_j(\overline{x}) &= 0, \quad j \in J_0(\overline{x}) \end{aligned} \tag{3}$$

with (unique) multipliers $\overline{\mu}_j \geq 0, j \in J_0(\overline{x})$ where L denotes the Lagrange function $L(x, \mu) = f(x) - \sum_{j \in J_0(\overline{x})} \mu_j g_j(x)$. By putting $G = (g_j, \ j \in J_0(\overline{x}))$, the Jacobian at a solution $(\overline{x}, \overline{\mu})$ reads:

$$J(\overline{x}, \overline{\mu}) = \begin{pmatrix} \nabla_x^2 L(\overline{x}, \overline{\mu}) & \nabla^T G(\overline{x}) \\ \nabla G(\overline{x}) & 0 \end{pmatrix}$$

As before, the Newton method applied to (3) (locally) has quadratic convergence if the Jacobian $J(\overline{x},\overline{\mu})$ is regular at a solution $(\overline{x},\overline{\mu})$. Introducing the

Problem set $\mathcal{P} = \{P = (f, g_1, \ldots, g_q)\} \equiv C^\infty(\mathbb{R}^n, \mathbb{R})^{1+q}$,

the following result states that the Newton method is genericly applicable.

Theorem 2. *(Jongen/Jonker/Twilt see [8]) There is an open and dense subset $\mathcal{P}_0 \subset \mathcal{P}$ such that for all $P \in \mathcal{P}_0$, LICQ holds at each feasible x and at each solution $(\overline{x},\overline{\mu})$ of (3) the Jacobian $J(\overline{x},\overline{\mu})$ is regular.*

We now come back to semi-infinite optimization. Let $\overline{x} \in \mathcal{F}$ be a feasible point of GSIP. We define the *active index set*

$$Y_0(\overline{x}) = \{\overline{y} \in Y(\overline{x}) \mid g(\overline{x},\overline{y}) = 0\}.$$

The following observation is crucial for this approach. By definition, any point $\overline{y}$ from $Y_0(\overline{x})$ is a (global) maximizer of the following parametric optimization problem

$$Q(\overline{x}): \qquad \min_y \; g(\overline{x},y) \quad \text{s.t.} \quad v_l(\overline{x},y) \le 0,\ l \in L. \tag{4}$$

(the so-called *lower level problem*). So any point $\overline{y} \in Y_0(\overline{x})$ must satisfy the F. John conditions,

$$\nabla_y \mathcal{L}^{(\overline{x},\overline{y})}(\overline{x},\overline{y},\overline{\gamma}) = 0, \quad \overline{\gamma}_0 + \sum_{l \in L_0(\overline{x},\overline{y})} \overline{\gamma}_l = 1, \tag{5}$$

with multipliers $0 \le \overline{\gamma}_0$, $0 \le \overline{\gamma} \in \mathbb{R}^{|L_0(\overline{x},\overline{y})|}$ where $L_0(\overline{x},\overline{y}) = \{l \in L \mid v_l(\overline{x},\overline{y}) = 0\}$ is the active index set and $\mathcal{L}^{(\overline{x},\overline{y})}$ the Lagrange function of $Q(\overline{x})$ (at $\overline{x}$, $\overline{y} \in Y_0(\overline{x})$),

$$\mathcal{L}^{(\overline{x},\overline{y})}(x,y,\gamma) = \gamma_0 g(x,y) - \sum_{l \in L_0(\overline{x},\overline{y})} \gamma_l v_l(x,y). \tag{6}$$

We will assume, that the following strong regularity conditions (of finite optimization) are satisfied for all points in $Y_0(\overline{x})$.

A1$_{red}$: *For any $\overline{y} \in Y_0(\overline{x})$ the following holds:*

1. *LICQ: $\nabla_y v_l(\overline{x},\overline{y})$, $l \in L_0(\overline{x},\overline{y})$ are linearly independent.*

2. *(Under LICQ there are unique multipliers $0 \le \overline{\gamma}_0$, $\overline{\gamma} \in \mathbb{R}_+^{|L_0(\overline{x},\overline{y})|}$ with $\overline{\gamma}_0 > 0$ and (5). We assume: $\overline{\gamma}_l > 0$, $l \in L_0(\overline{x},\overline{y})$ (strict complementary slackness).*

3. *The second order condition (SOC): With $\overline{\gamma}$ in 2.,*

$$\eta^T \nabla_y^2 \mathcal{L}^{(\overline{x},\overline{y})}(\overline{x},\overline{y},\overline{\gamma})\eta > 0, \quad \text{for all } \eta \in T(\overline{x},\overline{y}) \setminus \{0\}$$

where $T(\overline{x},\overline{y}) = \{\eta \in \mathbb{R}^m \mid \nabla_y v_l(\overline{x},\overline{y})\eta = 0,\ l \in L_0(\overline{x},\overline{y})\}$.

By using techniques from parametric optimization under this assumption the following can be proven.

Theorem 3. *Let at the feasible point $\overline{x}$ for GSIP the condition A_{red} be satisfied. Then*

(a) The active index set is finite, $Y_0(\overline{x}) = \{\overline{y}_1, \ldots, \overline{y}_r\}$, and there exists neighborhoods $U_{\overline{x}}$ of $\overline{x}$ and $V_{\overline{y}_j}$ of $\overline{y}_j$ and continuous mappings

$$y_j : U_{\overline{x}} \to V_{\overline{y}_j} \qquad \text{such that} \quad y_j(\overline{x}) = \overline{y}_j$$

such that for every $x \in U_{\overline{x}}$ the value $y_j(x)$ is the unique local solution of $Q(x)$.

(b) With the functions in (a) the following finite reduction holds: $x \in U_{\overline{x}} \cap \mathcal{F}$ is a local solution of GSIP if and only if $\overline{x}$ is a local solution of the so-called reduced problem

$$P_{red}(\overline{x}) : \quad \min_{x \in U_{\overline{x}}} f(x) \quad s.t. \quad G_j(x) := g(x, y_j(x)) \leq 0, \quad j = 1, \ldots, r\,.$$

(c) The functions $G_j(x) = g(x, y_j(x))$ are C^2-functions with derivatives

$$\nabla G_j(x) \quad = \quad \nabla_x \mathcal{L}^{(\overline{x}, \overline{y}_j)}(x, y_j(x), \gamma_j(x))$$

PROOF. (See [11] for details). Choose $j \in \{1, \cdots, r\}$ and put for brevity $(\overline{y}, \overline{\gamma}) = (\overline{y}_j, \overline{\gamma}_j)$. Assume $L_0(\overline{x}, \overline{y}) = \{1, \cdots, p\}$ and define $v := (v_1, \cdots, v_p)$. Consider the following Karush-Kuhn-Tucker equations for $Q(x)$, near $(\overline{x}, \overline{y}, \overline{\gamma})$, with Lagrange function $\overline{\mathcal{L}} = \mathcal{L}^{(\overline{x}, \overline{y}_j)}$:

$$F(x, y, \gamma) := \begin{array}{rcl} \nabla_y \overline{\mathcal{L}}(x, y, \gamma) & = & 0 \\ v(x, y) & = & 0 \end{array} \tag{7}$$

Under assumption LICQ and SOC of A_{red} the Jacobian $\nabla_{(y,\gamma)} F(\overline{x}, \overline{y}, \overline{\gamma})$ can be shown to be regular. The Implicit Function Theorem applied to (7) yields then the result (a) and (b). The derivative of $y(x)$ can be calculated by differentiating the identity $F(x, y(x), \gamma(x)) = 0$. We obtain for the second relation (the arguments $x, y(x), \gamma(x)$ omitted)

$$\nabla_x v + \nabla_y v \nabla y \quad = \quad 0 \tag{8}$$

For $G(x) = g(x, y(x))$ we obtain using $\nabla_y g \nabla y = \gamma^T \nabla_y v \nabla y = -\gamma^T \nabla_x v$ (cf. (7), (8))

$$\begin{aligned} \nabla G(x) &= \nabla_x g(x, y(x)) + \nabla_y g(x, y(x))\, \nabla y(x) \\ &= \nabla_x g(x, y(x)) - \gamma^T(x) \nabla_x v(x) = \nabla_x \overline{\mathcal{L}}(x, y(x), \gamma(x)) \end{aligned} \tag{9}$$

□

By this theorem locally near $\overline{x}$ the problem GSIP is equivalent to $P_{red}(\overline{x})$ which is a common finite optimization problem. Thus, the standard optimality conditions of finite optimization can be applied to obtain necessary optimality conditions for the semi-infinite problem P: If $\overline{x}$ is a local maximizer of GSIP then there are multipliers $\overline{\mu}_0 \geq 0$, $\overline{\mu}_j \geq 0, j = 1, \ldots, r$ (and multiplier vectors $[\overline{\gamma}_j]$ satisfying (5) for $\overline{y} = \overline{y}_j$) such that

$$\overline{\mu}_0 \nabla f(\overline{x}) - \sum_{j=1}^{r} \overline{\mu}_j \nabla_x \mathcal{L}^{(\overline{x}, \overline{y}_j)}(\overline{x}, \overline{y}_j, \overline{\gamma}_j) = 0\,, \quad \overline{\mu}_0 + \sum_{j=1}^{r} \overline{\mu}_j = 1\,. \tag{10}$$

This necessary condition even holds without assuming A_{red} (cf. [7]). Altogether we obtain the following complete system of optimality conditions (locally at a feasible $\overline{x}$ with $Y_0(\overline{x}) = \{\overline{y}_1, \ldots, \overline{y}_r\}$):

$$\mu_0 \nabla f(x) - \sum_{j=1}^{k} \mu_j \nabla_x \left(g(x, y_j) - \sum_{l \in L_0(\overline{x}, \overline{y}_j)} [\gamma_j]_l v_l(x, y_j) \right) = 0 \tag{11}$$

$$\mu_0 + \sum_{j=1}^{r} \mu_j = 1 \tag{12}$$

$$\forall\, j: \quad g(x, y_j) \overline{y}_j) \gamma_l^j v_l(x, y^j) = 0$$

and for $y_j \in Y(x), j = 1, \ldots, r$:

$$\gamma_0 \nabla_y g(x, y_j) - \sum_{l \in L_0(x, y_j)} [\gamma_j]_l \nabla_y v_l(x, y_j) = 0$$

$$\gamma_0 + \sum_{l \in L_0(x, y_j)} [\gamma_j]_l = 1$$

$$\forall\, l \in L_0(x, y_j): \quad v_l(x, y_j) = 0$$

Note that for common SIP-problems the system simplifies. Since the functions $v_l(y)$ does not depend on x in this case in the first equations the sum over $[\gamma_j]_l \nabla_x v_l(x, y_j)$ vanishes. As in finite optimization this system has $K = n + 1 + r(m+2) + \sum_{j=1}^{r} |L_0(\overline{x}, \overline{y}_j)|$ equations and equally many unknowns $x, \mu_j, y_j, [\gamma_j]$. So also here Newton-type methods may be applied to solve GSIP numerically (see [12]). As 'natural' regularity conditions we assume

RC: For each $x \in \mathcal{F}$ with any choice of solutions $[\gamma_j]$ of (10) for each $y_j \in Y_0(x)$ the following is true: If there is a solution μ of the relations (??) , (11) then

1. LICQ holds: $\nabla_x \mathcal{L}^{(x, y_j)}(x, y_j, \gamma_j)$, $y_j \in Y_0(x)$ are linearly independent.
2. For any $y_j \in Y_0(x)$ the assumption A_{red} holds.

For common SIP the following genericity result has been proven which gives the theoretical basis for the Kuhn-Tucker methods for solving SIP.

Theorem 4. (Jongen/Zwier [6]) *The set $\mathcal{P} = \{(f, g, v_l, l \in L)\} \equiv C^\infty(I\!R^n, I\!R) \times C^\infty(I\!R^{n+m}, I\!R)^{1+|L|}$ of all C^∞ SIP problems contains an open and dense subset $\mathcal{P}_0 \subset \mathcal{P}$ such that for all $P \in \mathcal{P}_0$ the regularity condition RC is satisfied.*

Unfortunately the situation is more complicated for GSIP where a genericity result as in Theorem 4 is not true. A counterexample is provided by the Example 2. Near the re-entrant corner point $\overline{x} = (0, 0)$ of $\mathcal{F}$ with corresponding active index point $\overline{y} = 0$ the system of optimality conditions can be written as:

$$\begin{aligned} 2\begin{pmatrix} x_1 \\ x_2 - 1 \end{pmatrix} + \mu\left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} - \gamma_1\begin{pmatrix} -1 \\ 0 \end{pmatrix} - \gamma_2\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right] &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ y - x_2 &= 0 \\ 1 - (\gamma_1 + \gamma_2) &= 1 \\ \gamma_1\,(y - x_1) &= 0 \\ \gamma_2\,(y + x_1) &= 0 \end{aligned}$$

The vector $(\overline{x}, \overline{y}, \overline{\mu}, \overline{\gamma}) = (0, 0, 0, 1, 1/2, 1/2)$ is a solution of this system. However the condition LICQ of A_{red} is not fulfilled.

On the other hand the Jacobian of the system can be shown to be regular at this solution $(\overline{x}, \overline{y}, \overline{\mu}, \overline{\gamma})$, *i.e.*, this point is an attraction point for the Newton method. But $\overline{x}$ is not a critical point in the common sense since there is a feasible direction $d = (\pm 1, 1)$ of descent: $\nabla f(\overline{x})d = -2 < 0$ (so $\overline{x}$ is not a candidate for a solution). In [10] it is conjectured that the genericity result will at least hold at all local solutions of GSIP (a proof is given for the linear case).

5 Problem under convexity conditions

In this section we will briefly show how certain convexity conditions on the SIP problem lead to a simpler structure both theoretically and numerically.

Let us assume that for each fixed x the function $g(x, \cdot)$ is convex in y and that the functions $v_l(x, y)$ are linear in y (*i.e.*, $Q(x)$ is a convex program). Then it is well-known that the KKT-system(with $\gamma_l \geq 0$)

$$\nabla_y g(x, y) - \sum_{l \in L_0(x,y)} \gamma_l \nabla_y v_l(x, y) = 0, \quad v_l(x, y) = 0, \forall\, l \in L_0(x, y),$$

is a necessary and sufficient condition for $y \in Y(x)$ to be a solution of $Q(x)$. If we introduce $v = (v_1, \ldots, v_k)$ and the corresponding multiplier vector $\gamma \in I\!R^k$ this KKT-system can equivalently be written in the *complementary* form:

$$\begin{aligned} \nabla_y g(x, y) - \nabla_y^T v(x, y)\gamma &= 0 \\ \gamma^T v(x, y) &= 0 \\ \gamma, -v(x, y) &\geq 0 \end{aligned}$$

So under these convexity assumptions the GSIP can equivalently be written as (non-linear) program with complementarity conditions:

$$\begin{aligned} \max_{x,y}\ f(x) \quad \text{s.t.} \qquad\qquad g(x,y) &\le 0 \\ \nabla_y g(x,y) - \nabla_y^T v(x,y)\gamma &= 0 \\ \gamma^T v(x,y_j) &= 0 \\ \gamma, -v(x,y_j) &\ge 0 \end{aligned}$$

In [10] *interior point techniques* are applied to solve GSIP numerically in this form.

References

[1] Dambrine M., Pierre M., *About stability of equilibrium shapes*, Mathem. Modelling and Num. Analysis Vol.34, N0 4, 811-834, (2000).

[2] Hettich R., Still G., *Semi-infinite programming models in Robotics*, in 'Parametric Optimization and Related Topics II', Guddat et al. (eds.), Akademie Verlag, Berlin (1991).

[3] Hettich R., Kortanek K., *Semi-infinite programming: Theory, methods and applications*, SIAM Review, vol 35, No.3, 380-429 (1993).

[4] Hettich R., Still G., *Second order optimality conditions for generalized semi-infinite programming problems*, Optimization Vol. 34, pp. 195-211 (1995).

[5] Hoffmann A. and Reinhardt R., *On reverse Chebyshev approximation problems*, Technical University of Ilmenau, Preprint No. M08/94 (1994).

[6] Jongen H. Th., Zwier G., *On the local structure of the feasible set in semi-infinite optimization*, in: Brosowski, Deutsch (eds.): Int. Ser. Num. Math. 72, Birkhäuser Verlag, Basel, 185-202, (1984).

[7] Jongen H. Th., Rückmann J.-J., Stein O., *Generalized semi-infinite optimization: A first order optimality condition and examples*, Mathematical Programming, Vol. 83, No. 1, 145-158 (1998).

[8] Jongen H.Th., Jonker P., Twilt F., Nonlinear Optimization in Finite Dimension, Kluwer Academic Publ., Dortrecht, (2000).

[9] Rückmann J.-J., Shapiro A., *First order optimality conditions in Generalized Semi-infinite Programming*, JOTA, Vol. 101, No. 2 (1999).

[10] Stein O., Still G., *Solving semi-infinite optimization problems with interior point techniques*, submitted.

[11] Still G., *Generalized semi-infinite programming: Theory and methods*, European Journal of Operational Research 119, 301-313, (1999).

[12] Still G., *Generalised semi-infinite programming: Numerical aspects*, Optimization 49, 223-242, (2001).

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