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THE CENTER CONDITIONS FOR A CUBIC SYSTEMS

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Abstract. In this paper we give conditions for system (2) to addmite (0,0) as a center.

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A cubic system with a singular point with pure imaginary eigenvalues ($\lambda_1 = \overline{\lambda}_2 = i$, $i^2 = -1$) by a nondegenerate transformation of variable and time rescaling can be brought to the form

$$\begin{aligned} \frac{dx}{dt} &= y + ax^2 + cxy + fy^2 + kx^3 + mx^2y + pxy^2 + ry^3, \\ \frac{dy}{dt} &= -(x + gx^2 + dxy + by^2 + sx^3 + qx^2y + nxy^2 + ly^3). \end{aligned} \tag{1}$$

The variables x, y and coefficients $a, b, \ldots, r, s$ are assumed to be real in (1). A singular point $(0,0)$ is a center or a focus for (1). The problem from distinguishing between a center and a focus, i.e. from finding the coefficient conditions on (1) under which $(0,0)$ is, for example, a center. These conditions are called the conditions for a center existence or the center conditions and the problem - the problem of the center.

Note that the singular point $(0,0)$ of the differential system (1) is called also weak focus (fine focus).

It is well known that the origin is a center for (1) if and only if all focal values g_{2j+1}, $j = \overline{1, \infty}$ vanish. The focal values are polynomials in coefficients of system (1). For example, the first of them looks as follows

$$g_3 = ac + 2ag + cf - bd - 2bf - dg + 3l + q - 3k - p.$$

If all the g_{2j+1} are zero up to $g_{2\tau+1}$, i.e. $g_{2j+1} = 0$, $j = \overline{1, \tau - 1}$ and $g_{2\tau+1} \neq 0$, then τ is called the order of the weak focus $(0,0)$.

It is known also that the system of differential equations (1) has a center at $(0,0)$ if and only if it has in some neighbourhood of the origin an independent of t holomorphic first integral $F(x,y) = C$ (an holomorphic integrating factor $\mu(x,y)$).

The problem of the center was solved for quadratic system ($k = l = m = n = p = q = r = s = 0$) by H. Dulac [6], and for symmetric cubic system ($a = b = c = d = f = g = 0$) by K.S. Sibirski [9].

If the cubic system (1) contains both quadratic and cubic nonlinearities the problem of the center is solved only in some particular cases (see, for example, [2, 3, 4, 5, 7, 8]). In this paper the problem is solved for cubic system

$$\begin{aligned} \dot{x} &= y + x^2 + (6b+5g)xy - 2y^2 + gx^3 + mx^2y - 5(b+g)xy^2 + y^3, \\ \dot{y} &= -(x + gx^2 + dxy + by^2 + qx^2y - (d+1)xy^2 - by^3). \end{aligned} \tag{2}$$

The system (2), after the change of coordinates $X = x/(1-y)$, $Z = y/(1-y)$, defines the following equation of nonlinear oscilations:

$$P_4(X)ZZ' = -XP_0(X) - 3XP_1(X)Z - P_2(X)Z^2 - P_3(X)Z^3, \tag{3}$$

where

$$\begin{aligned} P_0(X) &= \ 1 + gX, \ P_1(X) = (3 + d + (2g+q)X)/3, \\ P_2(X) &= \ b + (2+d)X + (g+q)X^2, \ P_3(X) = b, \\ P_4(X) &= \ 1 + (6b+5g)X + (m-d-1)X^2 - (g+q)X^3. \end{aligned}$$

The following proposition is the result of investigation.

Theorem 1. *The system (2) has a center at $(0,0)$ if and only if one of the following five conditions holds:*

1) $b = 0$, $q = g(d+1)$;
2) $d = 0$, $q = g$;
3) $d = -((5b+3g)(b+g)^2 + b + 2g)/(b+g)$,
   $m = ((5b+4g)(b+g)^2 + b - g)/(b+g)$,
   $q = -(b+g)(1 + (5b+3g)(b+g))$;
4) $g = -3b/2$, $d = -5$, $m = -7$, $q = b$;
5) $d = -5(b+g)(4 + (3b+2g)(4b+3g))/(13b+10g)$,
   $m = (5b(b+g)(3b+2g) - 21b + 30g)/(13b+10g)$,
   $q = -4g$, $\quad (3b+2g)(b+g)^2 + b - 2g = 0$.

Proof. Sufficiency. Consider the case 1). Assume that

$$\Big(500(d+2)g^4 + (88d^2 - 140dm + 212d + 25m^2 - 230m + 97)d^2 + 4(d-m+1)^3\Big)(d+1)(d+2) \neq 0. \tag{4}$$

Then the system (2) has Darboux first integral of the form $l_1^{\alpha_1} l_2^{\alpha_2} l_3^{\alpha_3} l_4^{\alpha_4} = const$, where $l_j = 1 + A_j x - y$, $j = 1, 2, 3$; $l_4 = 1 + (d+1)y$,

$$\alpha_1 = A_2 A_3 (A_2 - A_3)(4A_1 - A_2 - A_3)(A_1 + A_2 + A_3 + 5A_1 A_2 A_3),$$
$$\alpha_2 = A_1 A_3 (A_1 - A_3)(A_1 + A_3 - 4A_2)(A_1 + A_2 + A_3 + 5A_1 A_2 A_3),$$
$$\alpha_3 = A_1 A_2 (A_2 - A_1)(A_1 + A_2 - 4A_3)(A_1 + A_2 + A_3 + 5A_1 A_2 A_3),$$
$$\alpha_4 = (A_1 - A_2)(A_1 - A_3)(A_2 - A_3)(A_1 + A_2 + A_3)^2$$

and A_1, A_2, A_3, ($A_i \neq A_j$, $i \neq j$) are the roots of the algebraic equation $A^3 - 5gA^2 + (m - d - 1)A + g(d+2) = 0$. Since the center variety is closed in the space of coefficients of the system (2), then $(0,0)$ will be a center and in the case when the inequality (4) does not hold.

In the case 2) the system (2) has the integrating factor of the form $\mu = l_1^{\alpha_1} l_2^{\alpha_2} l_3^{\alpha_3}$, where $l_j = 1 + A_j x - y$, $j = 1, 2, 3$, A_j, $j = \overline{1,3}$ are the roots of the equation $A^3 - (6b + 5g)A^2 + (m-1)A + 2g$, and

$$\alpha_1 = (A_1^2(5A_2A_3 - 4) + 8A_1(A_2 + A_3) + 12)/(6(A_1 - A_2)(A_1 - A_3)),$$
$$\alpha_2 = (A_2^2(5A_1A_3 - 4) + 8A_2(A_1 + A_3) + 12)/(6(A_2 - A_1)(A_2 - A_3)),$$
$$\alpha_3 = (A_3^2(5A_2A_3 - 4) + 8A_3(A_1 + A_2) + 12)/(6(A_3 - A_1)(A_3 - A_2)).$$

Denote $\delta = b + g$. In the case 3) the system (2) has the integrating factor $\mu = l_1^{\alpha_1} l_2^{\alpha_2} l_3^{\alpha_3} l_4^{\alpha_4}$, where

$$l_1 = 1 + \delta x - y, \quad l_4 = 1 + (1 - b\delta^{-1})(\delta x - y),$$
$$l_{2,3} = 1 + \tfrac{1}{2}(4\delta + b \pm \sqrt{4\delta^3 + b^2\delta - 4b}/\sqrt{\delta})x - y,$$
$$\alpha_1 = 1, \quad \alpha_4 = (3\delta^3 + 2b\delta^2 + 2\delta - b)/(b - \delta),$$
$$\alpha_{2,3} = \tfrac{1}{2}(-3 \pm (6\delta^3 + 4b\delta^2 + 4\delta + 3b)\sqrt{\delta}/\sqrt{4\delta^3 + b^2\delta - 4b}).$$

Now we consider the cases 4) and 5). The substitution $Z = P_0(X)Y/(1 - P_1(X)Y)$ reduces the equation (3) to the form

$$Q_4(X)YY' = -X - Q_2(X)Y^2 - Q_3(X)Y^3,$$

where

$$\begin{aligned} Q_2(X) &\equiv P_0(X)P_2(X) - 3XP_1^2(X) + P_0'(X)P_4(X), \\ Q_3(X) &\equiv 2XP_1^3(X) - P_0(X)P_1(X)P_2(X) + P_0^2(X)P_3(X) + \\ &\quad P_0(X)P_1'(X)P_4(X) - P_0'(X)P_1(X)P_4(X), \\ Q_4(X) &\equiv P_0(X)P_4(X). \end{aligned}$$

By Theorem 9.4 of [1], if $Q_3(X) = X^{2j+1}\widetilde{P}(X)$, $\widetilde{P}(0) \neq 0$, then the origin is a center for the equation (3) if and only if the system of equations

$$\begin{aligned} y^4 R^3(x) Q_3^2(y) - x^4 R^3(y) Q_3^2(x) &= 0, \\ xQ(x)R^2(y) - yQ(y)R^2(x) &= 0, \end{aligned} \tag{5}$$

where

$$R(X) \equiv Q_4(X)[Q_3(X) - XQ_3'(X)] + 3XQ_2(X)Q_3(X),$$
$$Q(X) \equiv Q_4(X)[R'(X)Q_3(X) - 3R(X)Q_3'(X)] + 4Q_2(X)Q_3(X)R(X),$$

has in some neighborhood of $X = 0$ a holomorphic solution

$$Y = \phi(X), \quad \phi(0) = 0, \quad \phi'(0) = -1 \tag{6}$$

It is easy to verify that in the case 4) the equations (5) have a solution in the form of (6):

$$Y = \frac{3b^2X^2 - 20bX + 12 + (bX - 2)\sqrt{3(3b^2X^2 - 20bX + 12)}}{2b(2 - 3bX)},$$

and in the case 5), respectively the solution

$$Y = -\frac{3\delta^2X^2 + 10\delta X + 3 - (\delta X + 1)\sqrt{3(3\delta^2X^2 + 10\delta X + 3)}}{2(3\delta X + 1)}.$$

Necessity. We compute the first five focus quantities using the algorithm, described in [10]. The first one looks: $g_3 = q - g - d(b + g)$. From $g_3 = 0$ we find q:

$$q = g + d(b + g)$$

and substitute into the expression for g_5. We have $g_5 = bd(m - 5(b + g)(3b + 2g) - 2d - 3)$. If $b = 0$ then we have the case 1) and if $d = 0$, respectively the case 2).

Let

$$bd \neq 0 \tag{7}$$

and

$$m = 5(3b + 2g)(b + g) + 2d + 3.$$

The third focal value g_7 vanishes. The fourth focal value being canceled by non-zero factors is of the form $g_9 = f_1 f_2$, where

$$f_1 = (5b + 3g)(b + g)^2 + d(b + g) + b + 2g,$$
$$f_2 = 5(b + g)(4 + (3b + 2g)(4b + 3g)) + d(13b + 10g).$$

If $g + b = 0$, i.e. $g = -b$, then $g_9 = -3b^2d \neq 0$ (see (7)). Consider $b + g \neq 0$. Then $f_1 = 0$ implies the case 3). If the coefficient d in f_2 is equal to zero, i.e. $g = -13b/10$, then $f_2 = -3b(b^2 + 100)/50 \neq 0$ (see (7)). We require that $bdf_1(b + g)(13b + 10g) \neq 0$. From $f_2 = 0$ we express d:

$$d = -5(b + g)(4 + (3b + 2g)(4b + 3g))/(13b + 10g)$$

and substitute it in g_{11}. For g_{11}, after corresponding simplifications, i.e. after elimination of a denominator and non-zero factors, including numericale one, we have

$$g_{11} = (3b + 2g)((3b + 2g)(b + g)^2 + b - 2g).$$

If $3b + 2g = 0$ we have the case 4) and if $(3b + 2g)(b + g)^2 + b - 2g = 0$, respectively the case 5). □

The prof of the theorem 1 implies the following result.

Theorem 2. *The order of a weak focus for cubic differential system (2) is at most five.*

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