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BETA APPROXIMATING OPERATORS OF SECOND KIND FOR TWO VARIABLES

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Abstract: In this paper we present the Beta second - kind transform $T_{p,q,r,s}$ of a function $g : [0, +\infty) \times [0, +\infty) \to \mathbb{R}$ bounded and Lebesgue measurable in every interval $[a, b] \times [c, d]$, where $0 < a < b < \infty$ and $0 < c < d < \infty$, such that $T_{p,q,r,s} |g| < \infty$, by starting from the Beta distribution of second kind $b_{p,q}$.

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1.The Beta second-kind transform $\mathbf{T}_{p,q}$

In the paper [6] D.D.Stancu define the beta second-kind transform $T_{p,q}$ of a function $g \in M[0, +\infty]$, where $M[0, +\infty]$ is the linear space of functions $g(t)$, defined for $t \geq 0$, bounded and Lebesgue measurable in each interval $[a, b]$, $0 < a < b < \infty$. The beta distribution of second kind with the positive parameters p and q has the probability density

$$b_{p,q}(t) = \frac{t^{p-1}}{\beta(p,q)(1+t)^{p+q}} \tag{1}$$

when $t > 0$ and $b_{p,q}(t) = 0$ otherwise and $\beta(p, q)$ is the Beta function.

Using distribution (1) D.D. Stancu define the Beta second-kind transform $T_{p,q}$ of a function $g \in M[0, +\infty]$

$$T_{p,q}\, g = \int_0^{+\infty} g(t)\, b_{p,q}(t) dt = \frac{1}{\beta(p,q)} \int_0^{+\infty} g(t) \frac{t^{p-1}}{(1+t)^{p+q}}\, dt \tag{2}$$

such that $T_{p,q} |g| < \infty$.

2. The Beta second-kind transform $\mathbf{T}_{p,q,r,s}$ for functions of two variables

We consider the function $g(t, u)$, defined for $t \geq 0$ and $u \geq 0$, bounded and Lebesgue measurable in each interval $[a, b] \times [c, d]$, where $0 < a < b < \infty$ and $0 < c < d < \infty$. We denote by $M([0, +\infty) \times [0, +\infty))$ the linear space of such functions.

By using distribution (1) we define the beta second-kind transform $T_{p,q}$ of a function $g \in M([0, +\infty) \times [0, +\infty))$ by

$$T_{p,q,r,s}\, g = \int_0^{+\infty}\int_0^{+\infty} g(t,u)\, b_{p,q}(t)\, b_{r,s}(u)\, dt du =$$

$$= \frac{1}{\beta(p,q)\cdot\beta(r,s)} \int_0^{+\infty}\int_0^{+\infty} g(t,u) \frac{t^{p-1}}{(1+t)^{p+q}} \frac{u^{q-1}}{(1+u)^{r+s}}\, dt du \qquad (3)$$

We can observe that $T_{p,q,r,s}$ is a linear positive functional.

Applying the transform (3) to the Baskakov operator B_m for functions of two variables defined by

$$\left(B_m^t\, f\right)(t,u) = \sum_{k=0}^{\infty} \binom{m+k-1}{k} \frac{t^k}{(1+t)^{m+k}}\, f\left(\frac{k}{m}, u\right) \qquad (4)$$

$$\left(B_n^u\, f\right)(t,u) = \sum_{l=0}^{\infty} \binom{n+l-1}{l} \frac{u^l}{(1+u)^{n+l}}\, f\left(t, \frac{l}{n}\right)$$

we will prove the following theorem

Theorem 1. *The $T_{p,q,r,s}$ transform of Baskakov operator can be express under the form*

$$T_{p,q,r,s}\left(B_m^t B_n^u\, f\right)(t,u) = \sum_{k=0}^{\infty}\sum_{k=0}^{\infty} \binom{m+k-1}{k}\binom{n+l-1}{l} \cdot$$

$$\cdot \frac{p(q+1)\ldots(p+q-1)q(q+1)\ldots(q+m-1)}{(p+q)(p+q+1)\cdot\ldots\cdot(p+q+m+k-1)} \cdot$$

$$\cdot \frac{r(r+1)\ldots(r+s-1)s(s+1)\ldots(s+n-1)}{(r+s)(r+s+1)\cdot\ldots\cdot(r+s+n+l-1)} \cdot f\left(\frac{k}{m}, \frac{l}{n}\right) \qquad (5)$$

Proof. We have

$$\left(B_m^t B_n^u\, f\right)(t,u) = B_m^t\left(\sum_{l=0}^{\infty} \binom{n+l-1}{l} \frac{u^l}{(1+u)^{n+l}}\, f\left(t, \frac{l}{n}\right)\right) =$$

$$= \sum_{l=0}^{\infty} \binom{n+l-1}{l} \frac{u^l}{(1+u)^{n+l}} \left(B_m^t\, f\right)\left(t, \frac{l}{n}\right) =$$

$$= \sum_{l=0}^{\infty} \binom{n+l-1}{l} \frac{u^l}{(1+u)^{n+l}} \sum_{l=0}^{\infty} \binom{m+k-1}{l} \frac{t^k}{(1+t)^{m+k}} f\left(\frac{k}{m}, \frac{l}{n}\right) =$$

$$= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \binom{m+k-1}{k} \binom{n+l-1}{l} \frac{t^k}{(1+t)^{m+k}} \cdot \frac{u^l}{(1+u)^{n+l}} f\left(\frac{k}{m}, \frac{l}{n}\right) \tag{6}$$

$$T_{p,q,r,s}\left(B_m^t B_n^u f\right) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \binom{m+k-1}{k} \binom{n+l-1}{l} \cdot$$

$$\cdot \frac{1}{B(p,q)B(r,s)} \left[\int_0^{+\infty} \int_0^{+\infty} \frac{t^{p+k-1}}{(1+t)^{m+p+q+k}} \cdot \frac{u^{r+l-1}}{(1+u)^{r+s+n+l}} \, dt du \right] f\left(\frac{k}{m}, \frac{l}{n}\right) \tag{7}$$

In the integral $I = \int_0^{+\infty} \int_0^{+\infty} \frac{t^{p+k-1}}{(1+t)^{m+p+q+k}} \cdot \frac{u^{r+l-1}}{(1+u)^{r+s+n+l}} \, dt du$ we make the change of variables $y = \frac{t}{1+t}$, $z = \frac{u}{1+u}$ and we get

$$\begin{aligned} I &= \int_0^1 \int_0^1 y^{k+p-1}(1-y)^{m+q-1} z^{l+r-1}(1-z)^{n+s-1} dt du = \\ &= B(k+p, m+q) \cdot B(l+r, n+s) = \\ &= \frac{p(q+1)\dots(p+q-1)q(q+1)\dots(q+m-1)}{(p+q)(p+q+1)\cdot\dots\cdot(p+q+m+k-1)} \cdot \\ &\cdot \frac{r(r+1)\dots(r+s-1)s(s+1)\dots(s+n-1)}{(r+s)(r+s+1)\cdot\dots\cdot(r+s+n+l-1)} \cdot B(p,q) \cdot B(r,s). \end{aligned}$$

Then formula (7) becomes the formula (5).

If in (3) we choose $p = mx$, $q = m+1$, $r = ny$, $s = n+1$ then for any function $f \in M\left([0,+\infty) \times [0,+\infty)\right)$ we obtain the linear positive operator $L_{m,n}$ defined by

$$\begin{aligned} (L_{m,n}\, f\,)(x,y) &= T_{mx,m+1,ny,n+1} f(x,y) = \int_0^{+\infty} \int_0^{+\infty} f(t,u) b_{mx,m+1}(t) b_{ny,n+1}(u) dt du = \\ &= \frac{1}{B(mx,m+1) \cdot B(ny,n+1)} \int_0^{+\infty} \int_0^{+\infty} f(t,u) \frac{t^{mx-1}}{(1+t)^{mx+m+1}} \cdot \\ &\cdot \frac{u^{ny-1}}{(1+u)^{ny+n+1}} \, dt du. \end{aligned} \tag{8}$$

The operator $L_{m,n}$ reproduces the function $e_{00}(x,y)$ because

$$(L_{m,n}\,e_{00})(x,y)=\int_0^{+\infty}\int_0^{+\infty} b_{mx,m+1}(t)b_{ny,n+1}(u)\;dtdu=1.$$

For $e_{11}(x,y)=xy$ we start with

$$T_{p,q,r,s}e_{11}=\frac{1}{B(p,q)\,B(r,s)}\int_0^{+\infty}\int_0^{+\infty}\frac{t^p}{(1+t)^{p+q}}\cdot\frac{u^r}{(1+u)^{r+s}}dtdu.$$

We make the change of integration variables $y=\frac{t}{1+t}$, $z=\frac{u}{1+u}$, and we get

$$\begin{aligned}T_{p,q,r,s}e_{11} &= \frac{1}{B(p,q)\,B(r,s)}\int_0^1\int_0^1 y^p(1-y)^{q-2}z^r(1-z)^{s-2}dydz=\\ &= \frac{1}{B(p,q)B(r,s)}B(p+1,q-1)\cdot B(r+1,s-1).\end{aligned}$$

Applying the relations

$$B(a,b)=\frac{b-1}{a+b-1}\;B(a,b-1);\,B(a+1,b)=\frac{a}{a+b}B(a,b)\quad. \tag{9}$$

for $a=p+1,\ b=q$ and $a=p,\ b=q$ we obtain

$$\begin{aligned}T_{p,q,r,s}e_{11} &= \frac{1}{B(p,q)B(r,s)}\,\frac{p}{q-1}B(p,q)\cdot\frac{r}{s-1}B(r,s)=\\ &= \frac{p}{q-1}\cdot\frac{r}{s-1},\end{aligned}$$

Then $(L_{m,n}e_{11})(x,y)=T_{mx,m+1;ny,n+1}e_{11}=\frac{mx}{m+1-1}\cdot\frac{ny}{n+1-1}=xy.$

For $e_{10}(x,y)=x$, and $e_{20}(x,y)=x^2$ we have

$$\begin{aligned}T_{p,q,r,s}e_{10} &= \frac{1}{B(p,q)B(r,s)}\int_0^{+\infty}\int_0^{+\infty}\frac{t^p}{(1+t)^{p+q}}\cdot\frac{u^r}{(1+u)^{r+s}}dtdu=\\ &= \frac{1}{B(p,q)\,B(r,s)}\int_0^1\int_0^1 y^p(1-y)^{q-2}z^{r-1}(1-z)^{s-1}dydz=\end{aligned}$$

$$\frac{1}{B(p,q)\,B(r,s)} \cdot B(p+1,q+1)B(r,s) =$$

$$= \quad \frac{1}{B(p,q)} \cdot \frac{p}{q-1} B(p,q) = \frac{p}{q-1}.$$

$$\left(L_{m,n}\ e_{10}\right)(x,y) = T_{mx,m+1;ny,n+1} e_{10} = \frac{mx}{m+1-1} = x.$$

$$\begin{aligned} T_{p,q,r,s} e_{10} &= \frac{1}{B(p,q)B(r,s)} \int_0^{+\infty}\int_0^{+\infty} \frac{t^p}{(1+t)^{p+q}} \cdot \frac{u^r}{(1+u)^{r+s}} dt du = \\ &= \frac{1}{B(p,q)\,B(r,s)} \int_0^1\int_0^1 y^{p+1}(1-y)^{q-3} z^{r-1}(1-z)^{s-1} dy dz = \\ &= \frac{1}{B(p,q)\,B(r,s)} \cdot B(p+2,q-2)B(r,s) == \frac{p(p+1)}{(q-1)(q-2)}. \end{aligned}$$

$$\begin{aligned} \left(L_{m,n}\ e_{20}\right)(x,y) &= T_{mx,m+1;ny,n+1} e_{20} = \frac{mx(mx+1)}{m(m-1)} = \\ &= \frac{x(mx+1)}{m-1} = x^2 + \frac{x(x-1)}{m-1}. \end{aligned}$$

In the same way for $e_{01}(x,y) = y$ we have $\left(L_{m,n}\ e_{01}\right)(x,y) = y$ and for $e_{02}(x,y) = y^2$ we have $\left(L_{m,n}\ e_{02}\right)(x,y) = y^2 + \frac{y(y-1)}{n-1}$.

Theorem 2. *If $f \in C([0,+\infty) \times [0,+\infty))$ such that $L_{m,n}(|f|;x,y) < \infty$ then for any $m > 1$ and $n > 1$ we have*

$$\begin{aligned} \left|\left(L_{m,n}\ f\right)(x,y) - f(x,y)\right| &\le \left(1+\sqrt{x(x+1)}\right)\left(1+\sqrt{y(y+1)}\right) \cdot \\ &\quad \omega\left(f; \frac{1}{\sqrt{m+1}}, \frac{1}{\sqrt{n+1}}\right), \end{aligned}$$

where $\omega(f,\delta_1,\delta_2)$ represents the first order modulus of smoothness.

Proof. We use the following inequality given by D.D. Stancu in paper [3]

$$\left|\ f(x) - (L_m f)(x)\right| \le \left(1 + \frac{1}{\delta}\sigma_m(x)\right)\omega(f;\delta)$$

where $\sigma_m^2(x) = L_m((t-x)^2;x) = \frac{x+1}{m-1}$.

Then the form of this inequality for bivariable functions is following:

$$|(L_{m,n} f)(x,y) - f(x,y)| \leq$$

$$\leq \left(L_{m,n}(1;x,y) + \frac{1}{\delta_1}\sqrt{L_{m,n}(1;x,y)L_{m,n}((t-x)^2;x,y)} + \right.$$
$$+\frac{1}{\delta_2}\sqrt{L_{m,n}(1;x,y)L_{m,n}((u-y)^2;x,y)} +$$
$$\left. +\frac{1}{\delta_1\delta_2}\sqrt{L_{m,n}((t-x)^2;x,y)\cdot L_{m,n}((u-y)^2;x,y)}\right)\cdot$$
$$\cdot\omega(\delta_1,\delta_2) + |f(x,y)|\cdot|L_{m,n}(1;x,y)-1|\cdot$$
$$\cdot\left(1+\frac{1}{\delta_1}\sqrt{\frac{x(x-1)}{m-1}}+\frac{1}{\delta_2}\sqrt{\frac{y(y-1)}{n-1}}+\right.$$
$$\left.+\frac{1}{\delta_1\delta_2}\sqrt{\frac{x(x-1)}{m-1}}\cdot\sqrt{\frac{y(y-1)}{n-1}}\right)\omega(f;\delta_1,\delta_2),$$

because and $L_{m,n}\left((u-y)^2;x,y\right) = \frac{y(y+1)}{n-1}$

If we take $\delta_1 = \frac{1}{\sqrt{m-1}}$, $\delta_2 = \frac{1}{\sqrt{n-1}}$ we have inequality (11).

Corollary. If $f \in ([0,+\infty) \times [0,+\infty))$ and is continuous at all points of an interval $[a,b] \times [c,d]$ $(0 \leq a < b < \infty,\ 0 \leq c < d < \infty)$, then $L_{m,n}f$ converges uniformly in $[a,b] \times [c,d]$ to the function f when $m \to \infty$ and $n \to \infty$.

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