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ON UNIFORM OBSERVABILITY OF LINEAR DISCRETE-TIME STOCHASTIC SYSTEMS

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Abstract. We consider the linear discrete-time systems with independent random pertubations in Hilbert space. We obtain a necessary and sufficient conditions for the uniform exponential stability of the uniformly observable systems in terms of Lyapunov equations

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1.Preliminaries

Let H, V be separable real Hilbert spaces and let denote by $L(H,V)$ the Banach space of all bounded linear operators which transform H into V. If $H = V$ we put $L(H,V) = L(H)$. We write $\langle .,. \rangle$ for the inner product and $\|.\|$ for norms of elements and operators. If $A \in L(H)$ then A^* is the adjoint operator of A. The operator $A \in L(H)$ is said to be nonnegative and we write $A \geq 0$, if A is self-adjoint and $\langle Ax, x \rangle \geq 0$ for all $x \in H$. We denote by $\mathcal{H}$ the Banach subspace of $L(H)$ formed by all self-adjoint operators, by $\mathcal{K}$ the cone of all nonnegative operators of $\mathcal{H}$ and by I the identity operator on H.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and ξ be a real valued random variable on Ω. We write $E(\xi)$ for mean value (expectation) of ξ.

Let us consider the stochastic system $\{A; B\}$

$$x_{n+1} = A_n x_n + \xi_n B_n x_n, \quad x_k = x, k \in \mathbf{N}, \tag{1}$$

where $n \in \mathbf{N}$, $n \geq k$ ($\mathbf{N}$ is the set of all natural numbers), A_n, $B_n \in L(H)$ and $\xi_n, n \in \mathbf{N}$ are real independent random variables, which satisfy the conditions $E(\xi_n) = 0$ and $E|\xi_n|^2 = b_n < \infty$.

The random evolution operator associated to (1) is the operator $X(n,k)$ $n \geq k \geq 0$, where $X(k,k) = I$ and $X(n,k) = (A_{n-1} + \xi_{n-1} B_{n-1}) ... (A_k + \xi_k B_k)$, for all $n > k$. If $x_n(k,x)$ is the solution of (1) then $x_n(k,x) = X(n,k)x$.

Lemma 1. *(see [5]) Let $T \in L(\mathcal{H})$. If $T(\mathcal{K}) \subset \mathcal{K}$ then $\|T\| = \|T(I)\|$, where I is the identity operator on H.*

Let us introduce the linear and bounded operator $Q_n:\mathcal{H}\to\mathcal{H}$

$$Q_n(S) = A_n^* S A_n + b_n B_n^* S B_n, \tag{2}$$

where A_n, B_n and b_n, $n \in N$ are defined as above. We also consider the linear and bounded operator $T(n,k) : \mathcal{H} \to \mathcal{H}$, $T(n,k) = Q_k Q_{k+1} ... Q_{n-1}$, for all $n-1 \geq k$ and $T(k,k) = I$, where $I \in L(\mathcal{H})$ is the identity operator.

Proposition 1. *([6]) If $X(n,k)$ is the random evolution operator associated to (1) then we have*

$$\langle T(n,k)(S)x, y\rangle = E\langle SX(n,k)x, X(n,k)y\rangle$$

for all $S \in \mathcal{H}$ and $x, y \in H$. More, $T(n,k)$ satisfies the hypothesis of Lemma 1.

2. Uniform exponential stability and uniform observability

Definition 1. *The system $\{A; B\}$ is uniformly exponentially stable iff there exist $\beta \geq 1$, $a \in (0,1)$ such that we have*

$$E\left\|X(n,k)x\right\|^2 \leq \beta a^{n-k}\left\|x\right\|^2$$

for all $n \geq k \geq 0$ and $x \in H$.

If $C_n \in L(H,V)$, $n \in \mathbf{N}$ we denote by $\{A, C; B\}$ the system formed by the stochastic system $\{A; B\}$ and the observation relation $z_n = C_n x_n$.

Definition 2. *(see Definition 6 in [3]) The system $\{A, C; B\}$ is uniformly observable if there exist $n_0 \in \mathbf{N}$ and $\rho > 0$ such that*

$$\sum_{n=k}^{k+n_0} E\left\|C_n X(n,k)x\right\|^2 \geq \rho\left\|x\right\|^2 \tag{3}$$

for all $k \in N$ and $x \in H$.

Proposition 2. *(see [6]) The following statements are equivalent:*

a) the system (1) is uniformly exponentially stable;

b) there exist $\beta \geq 1$, $a \in (0,1)$ such that we have

$$\|T(n,k)\| \leq \beta a^{n-k} \text{ for all } n \geq k \geq 0.$$

Proposition 3. *(see [6]) The system $\{A, C; B\}$ is uniformly observable if there exist $n_0 \in \mathbf{N}$ and $\rho > 0$ such that*

$$\sum_{n=k}^{k+n_0} T(n,k)(C_n^* C_n) \geq \rho I \tag{4}$$

for all $k \in \mathbf{N}$.

2.1. The uniform exponential stability of uniformly observable systems and the Lyapunov equations

Consider the Lyapunov equation

$$P_n = Q_n(P_{n+1}) + F_n, n \in \mathbf{N}, \tag{5}$$

where Q_n is given by (2) and $F_n \in \mathcal{K}$ for all $n \in \mathbf{N}$. We need the following lemma.

Lemma 2. *a) The equation (5) has a nonnegative solution if and only if the series* $\sum_{k=n}^{\infty} T(k,n)(F_k)$ *converges in the strong operator topology for every* $n \geq 0$.

b) The equation (5) has a bounded nonnegative solution on $\mathbf{N}$ *if and only if there exists* $\alpha > 0$ *such that* $\sum_{k=n}^{\infty} T(k,n)(F_k) \leq \alpha I$, $n \geq 0$. *If the last inequality holds then* $W_n = \sum_{k=n}^{\infty} T(k,n)(F_k)$ *is a bounded nonnegative solution of (5).*

Proof. a) " $\Rightarrow$ " If the equation (5) has a nonnegative solution P_n then it follows

$$P_k = T(n+1,k)(P_{n+1}) + \sum_{l=k}^{n} T(l,k)(F_l) \tag{6}$$

for all $n \geq k$. Therefore $W_n = \sum_{l=k}^{n} T(l,k)(F_l) \leq P_k$ for all $n \geq k$. Consequently the increasing sequence $\{W_n\}_{n\geq k}$ is bounded above and $\{W_n\}_{n\geq k}$ converges in the strong operator topology to the operator $W \leq P_k$, $W = \sum_{l=k}^{\infty} T(l,k)(F_l)$.

" $\Leftarrow$ " If the series $\sum_{k=n}^{\infty} T(k,n)(F_k)$ converges in the strong operator topology then, for all $x \in H$, we have $\left\langle Q_n\left(\sum_{k=n+1}^{\infty} T(k,n+1)(F_k)\right)x, x\right\rangle + \langle F_n x, x\rangle =$

$$\left\langle \sum_{k=n+1}^{\infty} T(k,n+1)(F_k)A_n x, A_n x\right\rangle + b_n\left\langle \sum_{k=n+1}^{\infty} T(k,n+1)(F_k)B_n x, B_n x\right\rangle =$$

$$+\langle F_n x, x\rangle = \sum_{k=n+1}^{\infty} \langle Q_n(T(k,n+1)(F_k))x, x\rangle + \langle F_n x, x\rangle$$

$$= \sum_{k=n}^{\infty} \langle (T(k,n)(F_k))x, x\rangle = \left\langle \sum_{k=n}^{\infty} T(k,n)(F_k)x, x\right\rangle$$

Thus $\sum_{k=n}^{\infty} T(k,n)(F_k)$ is a nonnegative solution of (5). b) " $\Rightarrow$ " If the equation (5)

b) " ⇒ " If the equation (5) has a nonnegative and bounded solution $P_n(P_n \leq \alpha I, n \in \mathbb{N})$ then, repeating a similar procedure as above, we deduce $\sum_{l=k}^{\infty} T(l,k)(F_l) \leq P_k \leq \alpha I$. We obtain the conclusion.

" ⇐ " The converse and the last statement of b) follows from the hypothesis and by a).

We assume that the sequence C_n is bounded on $\mathbf{N}$, i.e. there exists a positive constant $\tilde{C}$ such that $\|C_n\| \leq \tilde{C}$ for all $n \in \mathbf{N}$ and we consider the Lyapunov equation

$$P_n = Q_n(P_{n+1}) + C_n^* C_n, n \in \mathbf{N} \tag{7}$$

Theorem 1. *If the system $\{A, C; B\}$ is uniformly observable then the following statements are equivalent:*

a) the system (1) is uniformly exponentially stable;

b) the equation (7) has a bounded nonnegative solution;

c) the equation (7) has a unique bounded nonnegative solution $P = (P_n)_{n \in \mathbf{N}}$ which has the property that there exist the positive constants m, M such as

$$m\|x\|^2 \leq \langle P_n x, x\rangle \leq M\|x\|^2 \tag{8}$$

for all $n \in \mathbf{N}$ and $x \in H$.

Proof.a) ⇒ *b*"Since $\sum_{k=n}^{\infty} \|T(k,n)(C_k^* C_k)\| \leq |\sum_{k=n}^{\infty} \|T(k,n)\| \, \|C_k^* C_k)\|$ we deduce from the hypothesis and the Proposition 5 that

$$\sum_{k=n}^{\infty} \|T(k,n)(C_k^* C_k)\| \leq |\sum_{k=n}^{\infty} \beta a^{n-k} \|C_k^* C_k)\| \leq \tilde{C}^2 \frac{\beta}{1-a}.$$

So, it follows that $\sum_{k=n}^{\infty} T(k,n)(C_k^* C_k)$ converges in the uniform operator topology and $\sum_{k=n}^{\infty} T(k,n)(C_k^* C_k) \leq \tilde{C}^2 \frac{\beta}{1-a} I$. By using Lemma 7 (b) we obtain the conclusion.

"b)⇒a" From Lemma 7 (b) we deduce that $W_n = \sum_{k=n}^{\infty} T(k,n)(C_k^* C_k)$ is a bounded on $\mathbb{N}$ nonnegative solution of the equation (7). Let us consider n_0, ρ the constants introduced by Definition 4. Since the system $\{A, C; B\}$ is uniformly observable we deduce from the boundedness of W_n that there exists $\alpha \geq 0$ such as

$$\alpha I \geq W_n \geq \rho I \tag{9}$$

for all $n \in \mathbb{N}$. It is not difficult to see that $Q_k(\mathcal{K})$ for all $k \in \mathbb{N}$. Thus, if L_p, $p \in \mathbb{N}$ is an increasing sequence, which converges in the strong operator topology to the operator

L then $Q_k(L_p)$ is an increasing bounded above sequence. As $p \to \infty$, $Q_k(L_p) \to Q_k(L)$, in the strong operator topology.

From the last statements and the definition of the operator $T(n,k)$ follows

$$T(n+n_0+1,n)(W_{n+n_0+1}) = \lim_{p\to\infty} T(n+n_0+1,n) \sum_{k=n+n_0+1}^{p} T(k,n)(C_k^*C_k) =$$

$$= \lim_{p\to\infty} \sum_{k=n+n_0+1}^{p} T(k,n)(C_k^*C_k) = \sum_{k=n+n_0+1}^{\infty} T(k,n)(C_k^*C_k).$$

We obtain $T(n+n_0+1,n)(W_{n+n_0+1}) \sum_{k=n+n_0+1}^{\infty} T(k,n)(C_k^*C_k) =$

$$W_n - \sum_{k=n}^{n+n_0} T(k,n)(C_k^*C_k) \le W_n - \rho I_n.$$

From (9) we have $T(n+n_0+1,n)(W_{n+n_0+1}) \le W_n - \frac{\rho}{\alpha} W_n$.

Thus $T(n+n_0+1,n)(W_{n+n_0+1}) \le qW_n$, where $q \in [0,1)$. We can assume, without to lose the generality that $q \in (0,1)$. Let $m \in \mathbb{N}$. Then there exists $c, r \in \mathbb{N}$ such as $m = (n_0+1)c + r$, $0 \le r \le n_0$. By induction and by the last inequality we obtain $T(m,n)(W_m) \le q^c T(n+r,n)(W_{n+r}) \le q^c W_n$.

From (9) and since $T(n,k)(\mathcal{K}) \subset \mathcal{K}$ for all $0 \le k < n$ it follows

$$\rho T(m,n)(I) \le q^{\frac{m-n}{n_0+1}} q^{\frac{-n_0}{n_0+1}} \alpha I.$$

By taking $\beta = q^{\frac{-n_0}{n_0+1}} \alpha/\rho$, $\alpha = q^{\frac{1}{n_0+1}}$ and by using Lemma 1, Proposition 5 and Definition 3 the conclusion follows. Because the implication "c)⇒b)" is obviously true, we only prove the implication "a)⇒c)". From "a)⇒b)" and since the system $\{A, C; B\}$ is uniformly observable it follows that $\sum_{k=n}^{\infty} T(k,n)(C_k^*C_k)$ is a bounded nonnegative solution of the equation (7), which satisfies the condition (8).Now we prove the uniqueness. Assume that P_n and R_n are two solutions of the equation (7) which satisfy (8). We have successively

$P_n - R_n = Q_n(P_{n+1} - R_{n+1})$, $P_n - R_n = T(n+k,n)(P_{n+k} - R_{n+k})$.

According to (8) we have $\|P_{n+k} - R_{n+k}\| \le \|P_{n+k}\| + \|R_{n+k}\| \le 2M$ and

$$\|P_n - R_n\| \le \|T(n+k,n)\| \, \|P_{n+k} - R_{n+k}\| \le 2M \, \|T(n+k,n)\| \tag{10}$$

By using the hypothesis and the Proposition 5 it follows $\lim_{k\to\infty} \|T(n+k,n)\| = 0$ for all $n \in \mathbb{N}$. By taking the limit in (10), as $k \to \infty$ we deduce $P_n = R_n$ for all $n \in \mathbb{N}$. The proof is complete.

2.2. The time invariant case

In this section we assume that $A_n = A$, $B_n = B$, $b_n = b$ and $C_n = C$ for all $n \in \mathbb{N}$.

Let us consider the algebraic Lyapunov equation

$$P = Q(P) + C^*C \tag{11}$$

The following corollary gives a characterization of the uniform exponential stability of the uniformly observable systems in terms of Lyapunov equations.

Corollary 1. *If the system $\{A, C; B\}$ is uniformly observable then (1) is uniformly exponentially stable if and only if the equation (11) has a unique positive solution.*

Proof. " $\Rightarrow$ " Since, in this case, the sequence $\{C_n^* C_n\}_{n\in\mathbf{N}}$ is constant ($C_n^* C_n = C^* C$ for all $n \in \mathbf{N}$) then it is bounded and we use Theorem 8 and Lemma 7 b) to deduce that $P_n = \sum_{k=n}^{\infty} T(k,n)(C^*C)$ is a bounded solution on $\mathbf{N}$ of the equation (7). From Proposition 5 it follows that the operator $I - Q$ is invertible and consequently $P_n = \sum_{k=0}^{\infty} Q^k(C^*C) = (I-Q)^{-1}(C^*C)$. Thus $P_n \stackrel{\text{not}}{=} P$, which does not depend on n, is a solution of the algebraic equation (11). More, since $\{A, C; B\}$ is uniformly observable we deduce that (8) holds and therefore P is positive. To show the uniqueness let P_1 be another positive solution of (11). Then P_1 is also a solution of the equation (7), which satisfies (8) and applying Theorem 8 it follows $P_1 = P$.

" $\Leftarrow$ " If (11) has a unique positive solution then it follows that this solution is also a bounded solution of (7). From Theorem 8 we obtain the conclusion. ■

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