CARPATHIAN J. MATH.
19 (2003), No. 1, 1 - 6

Simultaneous approximation by Schurer type operators

Dan Bărbosu and Mihai Bărbosu

Abstract. Let p be a given non negative integer.
For any $m \in \mathbb{N}$, let $\widetilde{B}_{m,p} : C\big([0, 1 + p]\big) \to C\big([0, 1]\big)$

$$\left(\widetilde{B}_{m,p}f\right)(x) = \sum_{k=0}^{m+p} \widetilde{p}_{m,k}(x) f(k/m)$$

be the Schurer operators (introduced and studied by Schurer, K., in 1962). Some approximation properties were studied in ([4]).

In the present paper we study properties of simultaneous approximation for $\widetilde{B}_{m,p}$. As particular cases, we get similar properties for Bernstein operators.

1. Preliminaries

Let $p \geq 0$ be a given integer. In ([3]) were introduced the Schurer operators $\widetilde{B}_{m,p} : C\big([0, 1 + p]\big) \to C\big([0, 1]\big)$

$$\left(\widetilde{B}_{m,p}f\right)(x) = \sum_{k=0}^{m+p} \widetilde{p}_{m,k}(x) f(k/m), \tag{1.1}$$

where $\widetilde{p}_{m,k}(x) = \binom{m+p}{k} x^k (1 - x)^{m+p-k}$ are the fundamental Schurer polynomials (see also ([2])).

The operators (1.1) generalize the classical Bernstein operators, obtained from (1.1) when $p = 0$.

In ([4]) a convergence theorem for the sequence $\big\{\widetilde{B}_{m,p}f\big\}_{m\in\mathbb{N}}$ and estimations for the rate of convergence under different assumptions on the approximated function were established.

In this present paper we study the simultaneous approximation of a function $f \in C^j\big([0, 1 + p]\big)$ using the Schurer operators (1.1).

Received: 08.01.2003; In revised form: 11.12.2003
2000 *Mathematics Subject Classification.* 41A36, 41A28.
Key words and phrases. *Linear positive operators, simultaneous approximation, Schurer operators, Bernstein operators, first order modulus of smoothness, second order modulus of smoothnes.*

Section 2 provides properties in connection with the derivatives of $\widetilde{B}_{m,p}f$. As consequences of these properties we get formulas for the derivatives of $\widetilde{B}_{m,p}$ in terms of finite and divided differences of approximated function.

In Section 3, we establish a convergence theorem for the sequence $\{D^j \widetilde{B}_{m,p} f)\}_{m\in\mathbb{N}}$, where D^j denotes the j-th order differential operator.

2. The derivatives of $\widetilde{B}_{m,p}f$

In what follows we denote by $\widetilde{p}_{m,k}(x)$ the fundamental Schurer polynomials

$$\widetilde{p}_{m,k}(x) = \binom{m+p}{k} x^k (1-x)^{m+p-k} \tag{2.1}$$

Lemma 2.1. *The polynomials* (2.1) *verify*

$$\begin{aligned} D^1\big(\widetilde{p}_{m,k}\big)(x) &= (m+p)\big\{\widetilde{p}_{m-1,k-1}(x) - \widetilde{p}_{m-1,k}(x)\big\} \\ &= \frac{k-(m+p)x}{x(1-x)}\,\widetilde{p}_{m,k}(x) \end{aligned} \tag{2.2}$$

where $\widetilde{p}_{0,0}(x) := 1$, $\widetilde{p}_{s,-1}(x) = \widetilde{p}_{s,s+1}(x) := 0$, $s \in \mathbb{N}^*$.

Proof. The assertion follows from (2.1), by direct computation. □

Lemma 2.2. *The following equalities*

$$\begin{aligned} D^1\big(\widetilde{B}_{m,p}f\big)(x) &= (m+p)\sum_{k=0}^{m+p-1} \widetilde{p}_{m-1,k}(x)\Delta_{\frac{1}{m}} f\left(\frac{k}{m}\right) \\ &= \frac{m+p}{m}\sum_{k=0}^{m+p-1} \widetilde{p}_{m-1,k}(x)\left[\frac{k}{m}, \frac{k+1}{m}, f\right] \end{aligned} \tag{2.3}$$

hold.

Proof. From definition (1.1), we have

$$D^1\big(\widetilde{B}_{m,p}f\big) = \sum_{k=0}^{m+p} \frac{d}{dx}\big(\widetilde{p}_{m,k}\big)(x) f(k/m)\,.$$

Next, applying Lemma 2.1, yields

$$\begin{aligned} D^1\big(\widetilde{B}_{m,p}f\big)(x) &= (m+p)\Bigg\{\sum_{k=0}^{m+p} \widetilde{p}_{m-1,k-1}(x) f(k/m) \\ &\quad - \sum_{k=0}^{m+p} \widetilde{p}_{m-1,k}(x) f(k/m)\Bigg\} \end{aligned}$$

Because $\widetilde{p}_{m-1,0}(x) = 0$, the first sum of the right side of the above equality can be written in the form

$$\sum_{k=0}^{m+p} \widetilde{p}_{m-1,k-1}(x) f(k/m) = \sum_{k=0}^{m+p-1} \widetilde{p}_{m-1,k}(x) \big((k+1)/m\big) .$$

Because $\widetilde{p}_{m,m+p-1}(x) = 0$, the second sum can be written in the form

$$\sum_{k=0}^{m+p} \widetilde{p}_{m-1,k}(x) f(k/m) = \sum_{k=0}^{m+p-1} \widetilde{p}_{m-1,k}(x) f(k/m) .$$

Therefore

$$\begin{aligned} D^1\big(\widetilde{B}_{m,p} f\big)(x) &= (m+p) \sum_{k=0}^{m+p-1} \widetilde{p}_{m-1,k}(x) \left\{ f\big((k+1)/m\big) - f(k/m) \right\} \\ &= (m+p) \sum_{k=0}^{m+p-1} \widetilde{p}_{m-1,k}(x) \Delta_{\frac{1}{m}} f(k/m) , \end{aligned}$$

i.e. the first equality (2.3) is proved.
Taking into account that

$$\big[k/m, (k+1)/m; f\big] = m \Delta_{\frac{1}{m}} f(k/m)$$

we get

$$D^1\big(\widetilde{B}_{m,p} f\big)(x) = \frac{m+p}{m} \sum_{k=0}^{m+p-1} \widetilde{p}_{m-1,k}(x) \left[\frac{k}{m}, \frac{k+1}{m}; f \right]$$

and the proof ends. □

Theorem 2.1. *For any non-negative integer $j \le m+p$ the operators* (1.1) *verify*

$$D^j \left(\widetilde{B}_{m,p} f \right)(x) = m^{[j]} \sum_{k=0}^{m+p-j} \widetilde{p}_{m-j,k}(x) \Delta^j_{1/m} f(k/m), \tag{2.4}$$

where $m^{[j]} = m(m-1)\dots(m-j+1)$.

Proof. For $j = 1$, the equality holds from Lemma 2.2.
Suppose that (2.4) holds for $j := j-1$, i.e.

$$D^{j-1} \left(\widetilde{B}_{m,p} f \right)(x) = m^{[j-1]} \sum_{k=0}^{m-j+1} \widetilde{p}_{m-j+1}(x) \Delta^{j-1}_{1/m} f(k/m).$$

Applying Lemma 1.1, we get

$$\begin{aligned} D^j\big(\widetilde{B}_{m,p}f\big)(x) &= m^{[j-1]}\sum_{j=0}^{m+p-j+1}\frac{d}{dx}\big(\widetilde{p}_{m-j+1}(x)\big)\Delta^{j-1}_{1/m}f(k/m) \\ &= m^{[j]}\Bigg\{\sum_{k=0}^{m+p-j+1}\widetilde{p}_{m-j+1,k-1}(x)\Delta^{j-1}_{1/m}f(k/m) \\ &- \sum_{k=0}^{m+p-j+1}\widetilde{p}_{m-j+1,k}(x)\Delta^{j-1}_{1/m}f(k/m)\Bigg\} \end{aligned} \tag{2.5}$$

After some transformations, from (2.5) we get

$$\begin{aligned} D^j\big(\widetilde{B}_{m,p}f\big)(x) &= (m+p)^{[j]}\sum_{k=0}^{m+p-j}\widetilde{p}_{m-j,k}(x) \\ &\times \left\{\Delta^{j-1}_{1/m}f\left(\frac{k+1}{m}\right)-\Delta^{j-1}_{1/m}f\left(\frac{k}{m}\right)\right\} \\ &= (m+p)^{[j]}\sum_{k=0}^{m+p-j}\widetilde{p}_{m-j+1,k-1}(x)\Delta^{j}_{1/m}f(k/m)\,, \end{aligned}$$

i.e. the desired identity is proved by induction after j. □

Corollary 2.1. *The following*

$$\big(\widetilde{B}_{m,p}f\big)(x)=\sum_{j=0}^{m+p}\binom{m+p}{j}\Delta^{j}_{1/m}f(0)x^j \tag{2.6}$$

holds.

Proof. Applying the Taylor formula, we get

$$\big(\widetilde{B}_{m,p}f\big)(x)=\sum_{j=0}^{m+p}D^j\big(\widetilde{B}_{m,p}f\big)(0)x^j\,. \tag{2.7}$$

From Theorem 2.1, we have

$$D^j\big(\widetilde{B}_{m,p}f\big)(0)=(m+p)^{[j]}\sum_{k=0}^{m+p-j}\widetilde{p}_{m-j,k}(0)\Delta^{j}_{1/m}f(k/m). \tag{2.8}$$

But $\widetilde{p}_{m-j,0}(x)=1$ and for any $k\geq 1, \widetilde{p}_{m-j,k}(0)=0$.
Therefore

$$D^j\big(\widetilde{B}_{m,p}f\big)(0)=(m+p)^{[j]}\Delta^{j}_{1/m}f(0). \tag{2.9}$$

Using (2.7), (2.8) and (2.9) we arrive to the desired result (2.6). □

Corollary 2.2. *For any integer j satisfying $1 < j \leq m + p$, the operators* (1.1) *verify*
(2.10)
$$D^j\left(\widetilde{B}_{m,p}f\right)(x) = \frac{(m+p)^{[j]}}{m^j}\, j! \sum_{k=0}^{m+p-j} \widetilde{p}_{m-j,k}(x)\left[\frac{k}{m}\,\ldots, \frac{k+j}{m}\,; f\right].$$

Proof. The assertion follows from Theorem 2.1, taking into account of the well known relation between finite and divided differences
$$\left[\frac{k}{m}, \frac{k+1}{m}, \ldots, \frac{k+j}{m}\,; f\right] = \frac{m^j}{j!}\,\Delta^j_{1/m} f(k/m)\,.$$
□

3. Simultaneous approximation

The main result of this section is

Theorem 3.1. *For any $f \in C^j\big([0,1+p]\big)$, the sequence $\left\{D^j \widetilde{B}_{m,p} f\right\}_{m\in\mathbb{N}}$ converges to $f^{(j)}$, uniformly on $[0,1]$ $(1 < j \leq m+p)$.*

Proof. Using the symbol of Landau (see for example ([5])), the relation $\lim\limits_{m\to\infty} \mu\left(\dfrac{1}{m}\right) = 0$ can be expressed in the form $\mu\left(\frac{1}{m}\right) = o(1)$.
From the above remark, we can write
$$(m+p)^{[j]}\Delta^j_{1/m} f(k/m) = (m+p)^j\{1+o(1)\}\cdot j!\left[\frac{k}{m}, \ldots, \frac{k+j}{m}\,; f\right]. \tag{3.1}$$
Using the mean theorem for divided differences, from (3.1) it follows that exists $\xi_k \in \left]\dfrac{k}{m}, \dfrac{k+j}{m}\right[$ so that
$$(m+p)^{[j]}\Delta^j_{1/m} f(k/m) = \{1+o(1)\} f^{(j)}(\xi_k)\,. \tag{3.2}$$
On the other hand, we have
$$f^{(j)}(\xi_k) = \left\{f^{(j)}(\xi_k) - f^{(j)}(k/m)\right\} + f^{(j)}(k/m) = o(1) + f(k/m)\,,$$
because $\left|\xi_k - \dfrac{k}{m}\right| < \dfrac{1}{m}$ and $f \in C^j\big([0,1+p]\big)$. Therefore we get the identity
$$\begin{aligned}\{1+o(1)\}f^{(j)}(\xi_k) &= \{1+o(1)\}\left\{o(1) + f^{(j)}(k/m)\right\}\\ &= o(1) + \big(1+o(1)\big) f^{(j)}(k/m)\\ &= f^{(j)}(k/m) + o(1). \end{aligned} \tag{3.3}$$

From Theorem 2.1, taking into account that $\xi_k \in [k/m\,, k+j/m[$, we get

$$
\begin{aligned}
D^j\left(\widetilde{B}_{m,p}f\right)(x) &= \sum_{k=0}^{m+p-j} \widetilde{p}_{m-j,k}\left\{f^{(j)}(k/m)+o(1)\right\} \\
&= \left(\widetilde{B}_{m-j}f^{(j)}\right)(x)+o(1). \qquad (3.4)
\end{aligned}
$$

Because the sequence $\big(\widetilde{B}_{m-j,p}\,g\big)_{m\geq j}$ converges to g, uniformly on $[0,1]$ for any $g \in C\big([0,1+p]\big)$, from (3.4) we arrive to the desired result. □

Corollary 3.1. *Let* $B_m := \widetilde{B}_{m,0}$ *be the Bernstein operator. The sequence* $\left\{D^j(B_m f)\right\}_{m\in\mathbb{N}}$ *converges to* $f^{(j)}$*, uniformly on* $[0,1]$*, for any* $f \in C^j\big([0,1]\big)$.

Proof. We apply Theorem 3.1, for $p=0$. □

References

[1] Agratini, O., *Aproximare prin operatori liniari*, (Romanian), Presa Universitară Clujeană, Cluj-Napoca, 2000

[2] Bărbosu, D., Bărbosu, M., *Properties of the fundamental Schurer polynomials*, Proceed. of "3^{rd} International Conference on Applied Mathematics" (icam3), Baia Mare-Borşa, Romania, October 10-13, 2002, in: Bul.Ştiin. Univ. Baia Mare, Ser.B, Fasc. Matematică-Informatică, vol.XVIII, Nr.2, 2002, 133-136.

[3] Bărbosu, D., *Schurer-Stancu type operators*, to appear in Studia "Univ. Babeş-Bolyai"

[4] Schurer, F., *Linear positive operators in approximation theory*, Math. Inst. Techn. Univ. Delft: Report, 1962

[5] Stancu, D.D., Coman, Gh., Agratini, O., Trîmbiţaş, R., *Analiză Numerică şi Teoria Aproximării*, vol.I, Presa Universitară Clujeană, Cluj-Napoca, 2001 (Romanian).

North University of Baia Mare
Department of Mathematics and Computer Science
Victoriei 76, 430122, Baia Mare, Romania
E-mail address: dbarbosu@ubm.ro
E-mail address: danbarbosu@yahoo.com

SUNY
Department of Mathematics and Computer Science
3500 New Campus Dr. Brockport,
NY114420, USA
E-mail address: mbarbosu@brockport.edu