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On the approximation of fixed points of weak contractive mappings

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ABSTRACT. In this paper, the class of weak contractive type mappings, introduced in [3] and studied in [3] and [4] is compared to some other well known contractive type mappings in Rhoades' classification [24]. As corollaries of our main results, we obtain several convergence theorems for approximating fixed points by means of Picard iteration. These complete or extend the corresponding results in literature by providing error estimates, rate of convergence for the used iterative method as well as results concerning the data dependence of the fixed points.

1. INTRODUCTION

Having in view that many of the most important nonlinear problems of applied mathematics reduce to solving a given equation which in turn may be reduced to finding the fixed points of a certain operator, on the one hand, and the fact that contractive (Lipschitzian) type conditions arise naturally for many of these problems, on the other hand, the metrical fixed point theory has developed significantly in the second part of the XX^{th} century.

A lot of *metrical* fixed point theorems have been obtained, more or less important from a theoretical point of view, which establish usually the existence, or the existence and the uniqueness of fixed points for various contractive type mappings.

Among these fixed point theorems, only a few are important from a practical point of view, that is, they provide a *constructive* method for finding the fixed points and also offer information on the error estimate (the rate of convergence) and on the data dependence of the fixed points.

But, from a practical point of view it is important not only to know that the fixed point exists (and, possibly, is unique), but also to be able to construct that fixed point.

Starting from these numerical commands, in [1] and [2] the author surveyed some of the most important classes of contractive type mappings for which the fixed points can be obtained by constructive methods.

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Very recently in [3] and [4] we introduced and studied the class of weak contractions, that include important contractive type mappings for which the fixed point can be obtained by means of Picard iteration.

In this paper, in order to illustrate how large is the class of weak contractions, we compare it to other important classes of contractive type mappings, following Rhoades' classification [24], [26]. We complete the results in [3] and [4] by giving two theorems on the data dependence of the fixed points. Some convergence theorems for the Picard iteration in various classes of contractive type mappings, are also obtain as corollaries of the main results in [3] and [4].

2. Preliminaries

The classical Banach's contraction principle is one of the most useful results in fixed point theory. In a metric space setting its full statement is given by the next theorem.

Theorem B. *Let* (X, d) *be a complete metric space and* $T : X \longrightarrow X$ *a map satisfying*

$$d(Tx, Ty) \le a\, d(x, y)\,, \quad \text{for all}\, x, y \in X\,, \tag{2.1}$$

where $0 < a < 1$ *is constant. Then:*

$(p1)$ *T has a unique fixed point p in X;*

$(p2)$ *The Picard iteration* $\{x_n\}_{n=0}^{\infty}$ *defined by*

$$x_{n+1} = Tx_n\,, \quad n = 0, 1, 2, \dots \tag{2.2}$$

converges to p, for any $x_0 \in X$.

$(p3)$ *The following estimates hold*:

$$d(x_n, x^*) \le \frac{a^n}{1-a}\, d(x_0, x_1)\,, \quad n = 0, 1, 2, \dots \tag{2.3}$$

$$d(x_n, x^*) \le \frac{a}{1-a}\, d(x_{n-1}, x_n)\,, \quad n = 1, 2, \dots \tag{2.4}$$

$(p4)$ *The rate of convergence of Picard iteration is given by*

$$d(x_n, x^*) \le a\, d(x_{n-1}, x^*)\,, \quad n = 1, 2, \dots \tag{2.5}$$

Remarks.

A map satisfying $(p1)$ and $(p2)$ in Theorem B is said to be a *Picard operator*, see Rus [29], [31].

A mapping satysfying (2.1) is usually called *strict contraction* or *a-contraction.* Theorem B shows that any strict contraction is a Picard operator.

Theorem B has many applications in solving nonlinear equations. Its merit is not only to state the existence and uniqueness of the fixed point of the strict contraction T but also to show that the fixed point can be approximated by means of Picard iteration (2.2). Moreover, for this iterative method both a priori (2.3) and a posteriori (2.4) error estimates are available.

The inequality (2.5) shows that the rate of convergence of Picard iteration is linear in the class of strict contractions.

Despite these important features, Thoeorem B suffers from one drawback - the contractive condition (2.1) forces T be continuous on X.

It is then natural to ask if there exist contractive conditions which do not imply the continuity of T. This was answered in the affirmative by R. Kannan [14] in 1968, who proved a fixed point theorem which extends Theorem B to mappings that need not be continuous, by considering instead of (2.1) the next condition: there exists $b \in \left(0, \frac{1}{2}\right)$ such that

$$d(Tx, Ty) \leq b\big[d(x, Tx) + d(y, Ty)\big], \quad \text{for all } x, y \in X\,. \tag{2.6}$$

Following the Kannan's theorem, a lot of papers were devoted to obtaining fixed point theorems for various classes of contractive type conditions that do not require the continuity of T, see for example, Rus [29], [32], Taskovic [34], and references therein.

One of them, actually a sort of dual of Kannan fixed point theorem, due to Chatterjea [5], is based on a condition similar to (2.6): there exists $c \in \left(0, \frac{1}{2}\right)$ such that

$$d(Tx, Ty) \leq c\big[d(x, Ty) + d(y, Tx)\big], \quad \text{for all } x, y \in X. \tag{2.7}$$

It is well known, see Rhoades [24], that the contractive conditions (2.1) and (2.6), as well as (2.1) and (2.7), respectively, are independent.

In 1972, Zamfirescu [35] obtained a very interesting fixed point theorem, by combining (2.1), (2.6) and (2.7).

Theorem Z. *Let (X, d) be a complete metric space and $T : X \longrightarrow X$ a map for which there exist the real numbers a, b and c satisfying $0 < a < 1$, $0 < b,\ c < 1/2$ such that for each pair x, y in X, at least one of the following is true:*

(z_1) $d(Tx, Ty) \leq a\, d(x, y)$;

(z_2) $d(Tx, Ty) \leq b\big[d(x, Tx) + d(y, Ty)\big]$;

(z_3) $d(Tx, Ty) \leq c\big[d(x, Ty) + d(y, Tx)\big]$.

Then T is a Picard operator.

One of the most general contraction condition for which the map satisfying it is still a Picard operator, has been obtained by Ciric [8] in 1974:

there exists $0 < h < 1$ such that

(2.8) $$d(Tx,Ty) \le h \cdot \max\left\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\right\}, \quad \text{for all } x, y \in X .$$

Remark. A mapping satisfying (2.8) is commonly called *quasi contraction.*

It is obvious that each of the conditions (2.6), (2.7) and (z_1)-(z_3) implies (2.8).

There exist many other fixed point theorems based on contractive conditions of this type: [5]-[11], [15]-[17], [19]-[22], [24], [26], [33], see also the monographs Berinde [1], [2], Rus [29], Taskovic [34].

In a recent paper [3] we introduced and studied a general class of contractive type mappings, whose members are called weak contractions. This class includes mappings satisfying the previous contractive conditions (except for quasi contractions, which are known to be only in part included in the class of weak contractions). The next two sections present the most significant results obtained in [3] and [4], respectively.

3. Weak contractions

Definition 1. [3] Let (X, d) be a metric space. A map $T : X \longrightarrow X$ is called *weak contraction* or *(δ, L)-contraction* if there exist a constant $\delta \in (0, 1)$ and some $L \ge 0$ such that

(3.1) $$d(Tx,Ty) \le \delta \cdot d(x,y) + Ld(y,Tx), \quad \text{for all } x, y \in X .$$

Remark 1. Due to the symmetry of the distance, the weak contraction condition (3.1) implicitly includes the following dual one

(3.2) $$d(Tx,Ty) \le \delta \cdot d(x,y) + L \cdot d(x,Ty), \quad \text{for all } x, y \in X ,$$

obtained from (3.1) by formally replacing $d(Tx, Ty)$ and $d(x, y)$ by $d(Ty, Tx)$ and $d(y, x)$, respectively, and then interchanging x and y.

Consequently, in order to check the weak contractiveness of T, it is necessary to check both (3.1) and (3.2).

Obviously, any strict contraction satisfies (3.1), with $\delta = a$ and $L = 0$, and hence is a weak contraction (that possesses a unique fixed point).

Other examples of weak contractions given in [3] are collected in the next proposition.

Proposition 1. *1) Any Kannan mapping, i.e. any mapping satisfying the contractive condition (2.6), is a weak contraction.*

2) Any mapping T satisfying the contractive condition (2.7) is a weak contraction.

3) Any Zamfirescu mapping, i.e., any mapping satisfying the assumptions in Theorem Z, is a weak contraction.

4) Any quasi contraction with $0 < h < 1/2$ *is a weak contraction.*

There are many other examples of contractive conditions which implies the weak contractiveness condition, see for example Taskovic [34], Rus [30] and Berinde [1]. Some of them will also be mentioned in Section 5.

The main results in [3] are given here as Theorem 1 (an existence theorem) and Theorem 2 (an existence and uniqueness theorem). Their main merit is that extend Theorems B and Z to the larger class of weak contractions, in the spirit of Theorem B, that is, in such a way that they offer a method for approximating the fixed point, for which both a priori and a posteriori estimates are available.

Theorem 1. *Let* (X, d) *be a complete metric space and* $T : X \longrightarrow X$ *a weak contraction. Then*

1) $F(T) = \{x \in X : Tx = x\} \neq \phi$*;*

2) *For any* $x_0 \in X$*, the Picard iteration* $\{x_n\}_{n=0}^{\infty}$ *given by* (1.2) *converges to some* $x^* \in F(T)$*;*

3) *The following estimates*

$$d(x_n, x^*) \leq \frac{\delta^n}{1-\delta}\, d(x_0, x_1)\,, \quad n = 0, 1, 2, \ldots$$

$$d(x_n, x^*) \leq \frac{\delta}{1-\delta}\, d(x_{n-1}, x_n)\,, \quad n = 1, 2, \ldots$$

hold, where δ *is the constant appearing in* (3.1).

Remark 2. 1) As resulting by Corollaries 1-3, Theorem 1 is a significant extension of Theorem B, Theorem Z and many other related results, as shown in Section 5.

2) Note that, although the fixed point theorems mentioned at 1) actually forces the uniqueness of the fixed point, the weak contractions need not have a unique fixed point, as shown by Example 1.

Recall, see Rus [31], [32], [40], that an operator $T : X \longrightarrow X$ is said to be a *weakly Picard operator* if the sequence $\{T^n x_0\}_{n=0}^{\infty}$ converges for all $x_0 \in X$ and the limits are fixed points of T. Therefore, Theorem 3.1 provides a large class of weakly Picard operators.

3) It is easy to see that condition (3.1) implies the so called Banach orbital condition

$$d(Tx, T^2x) \leq a\, d(x, Tx)\,, \quad \textit{for all}\; x \in X\,,$$

studied by various authors in the context of fixed point theorems, see for example Kasahara [38], Hicks and Rhoades [12], Ivanov [13], Rus [29], [39], [40, Example 4.6] and Taskovic [34].

As we have shown in [3], it is possible to force the uniqueness of the fixed point of a weak contraction, by imposing an additional contractive condition, quite similar to (3.1), as shown by the next theorem.

Theorem 2. *Let (X,d) be a complete metric space and $T : X \longrightarrow X$ a weak contraction for which there exist $\theta \in (0,1)$ and some $L_1 \geq 0$ such that*

$$d(Tx,Ty) \leq \theta \cdot d(x,y) + L_1 \cdot d(x,Tx)\,, \quad \text{for all } x,y \in X\,. \tag{3.3}$$

Then

1) *T has a unique fixed point, i.e., $F(T) = \{x^*\}$;*

2) *The Picard iteration $\{x_n\}_{n=0}^{\infty}$ given by (1.2) converges to x^*, for any $x_0 \in X$;*

3) *The a priori and a posteriori error estimates*

$$\begin{aligned} d(x_n,x^*) &\leq \frac{\delta^n}{1-\delta}\,d(x_0,x_1)\,, \quad n = 0,1,2,\ldots \\ d(x_n,x^*) &\leq \frac{\delta}{1-\delta}\,d(x_{n-1},x_n)\,, \quad n = 1,2,\ldots \end{aligned}$$

hold.

4) *The rate of convergence of the Picard iteration is given by*

$$d(x_n,x^*) \leq \theta\, d(x_{n-1},x^*)\,, \quad n = 1,2,\ldots$$

Remark 3. Note that, by the symmetry of the distance, (3.3) is satisfied for all $x,y \in X$ if and only if

$$d(Tx,Ty) \leq \theta\, d(x,y) + L_1 d(y,Ty)\,, \tag{3.4}$$

also holds, for all $x,y \in X$.

So, similarly to the case of the dual conditions (3.1) and (3.2), in concrete applications it is necessary to check that both conditions (3.3) and (3.4) are satisfied.

As shown in the proofs of Corollaries 1-3, an operator T satisfying one of the conditions (2.1), (2.6), (2.7), or the conditions in Theorem Z, also satisfies the uniqueness conditions (3.3) and (3.4). Therefore, in view of Example 1, Theorem 2 (and also Theorem 1) properly generalizes Theorem Z. Applying Theorem 1 we are able to prove the following extensions of Kannan, Chatterjea and Zamfirescu fixed point theorems.

Corollary 1. *Let (X,d) be a complete metric space and $T : X \longrightarrow X$ a Kannan operator, i.e., a mapping satisfying (2.6). Then*

1) *T has a unique fixed point, i.e., $F(T) = \{x^*\}$;*

2) *The Picard iteration $\{x_n\}_{n=0}^{\infty}$ given by (1.2) converges to x^*, for any $x_0 \in X$;*

3) *The a priori and a posteriori error estimates*

$$\begin{aligned} d(x_n,x^*) &\leq \frac{\delta^n}{1-\delta}\,d(x_0,x_1)\,, \quad n = 0,1,2,\ldots \\ d(x_n,x^*) &\leq \frac{\delta}{1-\delta}\,d(x_{n-1},x_n)\,, \quad n = 1,2,\ldots \end{aligned}$$

hold, where $\delta = \dfrac{b}{1-b}$.

4) *The rate of convergence of the Picard iteration is given by*

$$d(x_n, x^*) \le \delta\, d(x_{n-1}, x^*)\,, \quad n = 1, 2, \dots$$

Proof. By condition (2.6) and triangle rule, we get

$$\begin{aligned} d(Tx,Ty) &\le b\big[d(x,Tx)+d(y,Ty)\big] \le \\ &\le b\Big\{\big[d(x,y)+d(y,Tx)\big]+\big[d(y,Tx)+d(Tx,Ty)\big]\Big\} \end{aligned}$$

which yields

$$(1-b)d(Tx,Ty) \le bd(x,y) + 2b \cdot d(y,Tx)$$

and which implies

$$d(Tx,Ty) \le \frac{b}{1-b}\, d(x,y) + \frac{2b}{1-b}\, d(y,Tx)\,, \quad \text{for all } \ x, y \in X\,,$$

i.e., in view of $0 < b < \dfrac{1}{2}$, (3.1) holds with $\delta = \dfrac{b}{1-b}$ and $L = \dfrac{2b}{1-b}$.

Since (2.6) is symmetric with respect to x and y, (3.2) also holds.

In a similar way, by the same condition (2.6) and triangle rule, we get

$$\begin{aligned} d(Tx,Ty) &\le b\big[d(x,Tx)+d(y,Ty)\big] \le \\ &\le b\Big\{d(x,Tx)+\big[d(y,x)+d(x,Tx)+d(Tx,Ty)\big]\Big\} \end{aligned}$$

which yields

$$(1-b)d(Tx,Ty) \le bd(x,y) + 2b \cdot d(x,Tx)$$

and therefore

$$d(Tx,Ty) \le \frac{b}{1-b}\, d(x,y) + \frac{2b}{1-b}\, d(x,Tx)\,, \quad \text{for all } \ x, y \in X\,,$$

which shows that (3.3) and (3.4) hold with $\theta = \dfrac{b}{1-b}$ and $L_1 = \dfrac{2b}{1-b}$.

The conclusion now follows by applying Theorem 2. □

Corollary 2. *Let* (X, d) *be a complete metric space and* $T : X \longrightarrow X$ *a Chatterjea operator, i.e., a mapping satisfying (2.7). Then*

1) T *has a unique fixed point, i.e.,* $F(T) = \{x^*\}$*;*

2) *The Picard iteration* $\{x_n\}_{n=0}^{\infty}$ *given by* (1.2) *converges to* x^**, for any* $x_0 \in X$*;*

3) *The a priori and a posteriori error estimates*

$$\begin{aligned} d(x_n, x^*) &\le \frac{\delta^n}{1-\delta}\, d(x_0, x_1)\,, \quad n = 0, 1, 2, \dots \\ d(x_n, x^*) &\le \frac{\delta}{1-\delta}\, d(x_{n-1}, x_n)\,, \quad n = 1, 2, \dots \end{aligned}$$

hold, where $\delta = \dfrac{c}{1-c}$.

4) *The rate of convergence of the Picard iteration is given by*

$$d(x_n, x^*) \leq \delta\, d(x_{n-1}, x^*)\,, \quad n = 1, 2, \ldots$$

Proof. By (2.7) and $d(x, Ty) \leq d(x, y) + d(y, Tx) + d(Tx, Ty)$ we get after simple computations,

$$d(Tx, Ty) \leq \frac{c}{1-c}\, d(x, y) + \frac{2\,c}{1-c}\, d(y, Tx)\,,$$

which is (3.1), with $\delta = \dfrac{c}{1-c} < 1$ (since $c < 1/2$) and $L = \dfrac{2\,c}{1-c} \geq 0$. The symmetry of (2.7) also implies (2.2). In a similar way we prove that (3.3) and (3.4) are also satisfied. The conclusion now follows by Theorem 2. □

By Theorem B, Corollary 1 and Corollary 2 we get

Corollary 3. *Let* (X, d) *be a complete metric space and* $T : X \longrightarrow X$ *a Zamfirescu operator, i.e., a mapping satisfying the contractive conditions in Theorem Z. Then*

1) T *has a unique fixed point, i.e.,* $F(T) = \{x^*\}$*;*

2) *The Picard iteration* $\{x_n\}_{n=0}^{\infty}$ *given by* (1.2) *converges to* x^**, for any* $x_0 \in X$*;*

3) *The a priori and a posteriori error estimates*

$$\begin{aligned} d(x_n, x^*) &\leq \frac{\delta^n}{1-\delta}\, d(x_0, x_1)\,, \quad n = 0, 1, 2, \ldots \\ d(x_n, x^*) &\leq \frac{\delta}{1-\delta}\, d(x_{n-1}, x_n)\,, \quad n = 1, 2, \ldots \end{aligned}$$

hold, where $\delta = \max\{a, \dfrac{b}{1-b}, \dfrac{c}{1-c}\}$.

4) *The rate of convergence of the Picard iteration is given by*

$$d(x_n, x^*) \leq \delta\, d(x_{n-1}, x^*)\,, \quad n = 1, 2, \ldots$$

Corollary 4. *Let* (X, d) *be a complete metric space and* $T : X \longrightarrow X$ *a quasi contraction with* $0 < h < 1/2$*. Then*

1) T *has a unique fixed point, i.e.,* $F(T) = \{x^*\}$*;*

2) *The Picard iteration* $\{x_n\}_{n=0}^{\infty}$ *given by* (1.2) *converges to* x^**, for any* $x_0 \in X$*;*

3) *The a priori and a posteriori error estimates*

$$\begin{aligned} d(x_n, x^*) &\leq \frac{\delta^n}{1-\delta}\, d(x_0, x_1)\,, \quad n = 0, 1, 2, \ldots \\ d(x_n, x^*) &\leq \frac{\delta}{1-\delta}\, d(x_{n-1}, x_n)\,, \quad n = 1, 2, \ldots \end{aligned}$$

hold, where $\delta = \dfrac{h}{1-h}$.

4) *The rate of convergence of the Picard iteration is given by*

$$d(x_n, x^*) \leq \delta\, d(x_{n-1}, x^*)\,, \quad n = 1, 2, \ldots$$

Proof. Let $T : X \longrightarrow X$ be a quasi contraction with $0 < h < 1/2$, i.e. an operator satisfying

$$d(Tx, Ty) \leq h \cdot M(x, y), \text{for all}\, x, y \in X$$

where $0 < h < 1/2$ and

$$M(x, y) = \max\big\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\big\}.$$

Let $x, y \in X$ be arbitrary taken. We have to discuss five possible cases, but due to the symmetry of $M(x, y)$, it suffices to consider only three of them.

Case 1. $M(x, y) = d(x, y)$, when, in virtue of (2.8), condition (3.1) and (3.2) are obviously satisfied (with $\delta = h$ and $L = 0$).

Case 2. $M(x, y) = d(x, Tx)$, when by (2.8) and triangle rule we get

$$d(Tx, Ty) \leq h\, d(x, Tx) \leq h\big[d(x, y) + d(y, Tx)\big]\,,$$

and so (3.1) holds with $\delta = h$ and $L = h$.

Since $d(x, Tx) \leq d(x, Ty) + d(Ty, Tx)$, we get

$$d(Tx, Ty) \leq \frac{h}{1-h}\, d(x, Ty) \leq \delta d(x, y) + \frac{h}{1-h}\, d(x, Ty)\,,$$

for all $\delta \in (0, 1)$. So (3.2) also holds.

Case 3. $M(x, y) = d(x, Ty)$, when (3.2) is obviously true and (3.1) is obtained only if $h < \dfrac{1}{2}$.

Indeed, since by (2.8), $d(Tx, Ty) \leq h \cdot d(x, Ty)$ and

$$d(x, Ty) \leq d(x, y) + d(y, Tx) + d(Tx, Ty)\,,$$

one obtains

$$d(Tx, Ty) \leq \frac{h}{1-h}\, d(x, y) + \frac{h}{1-h}\, d(y, Tx)\,.$$

Therefore both (3.1) and (3.2) hold with

$$\delta = L = \max\{0, h, \frac{h}{1-h}\} = \frac{h}{1-h} < 1.$$

In a similar manner we prove that (3.3) and (3.4) are also satisfied with

$$\theta = L_1 = \max\{0, h, \frac{h}{1-h}\} = \frac{h}{1-h} < 1.$$

To obtain the conclusion we apply Theorem 2. □

The next theorem completes Theorem 1 (and hence Corollaries 1-4 as well) with a result concerning the data dependence of the fixed points in the class of weak contractions.

Theorem 3. *Let* (X, d) *be a complete metric space and* $T : X \longrightarrow X$ *a* (δ, L)*-weak contraction and* $S : X \longrightarrow X$ *be an approximate operator of* T*, i.e., an operator for which there exist* $\eta > 0$ *such that for all* $x \in X$ *we have*

$$d(Tx, Sx) \le \eta.$$

Let x_T *and* x_S *be fixed points of* T *and* S*, respectively. If* $\delta + L < 1$*, then*

$$d(x_T, x_S) \le \frac{\eta}{1 - \delta - L}.$$

Proof. We have

$$\begin{aligned} d(x_T, x_S) = d(Tx_T, Sx_S) &\le d(Tx_T, Tx_S) + d(Tx_S, Sx_S) \\ &\le \delta d(x_T, x_S) + L d(x_S, Tx_T) + \eta \end{aligned}$$

which implies the desired inequality. □

At the end of this section we present two of the examples given in [3]. Let $[0, 1]$ be the unit interval with the usual norm.

Example 1. Let $T : [0, 1] \longrightarrow [0, 1]$ be the identity map, i.e., $Tx = x$, for all $x \in [0, 1]$. Then

1) T does not satisfy the Ciric's contractive condition (2.8).

2) T satisfies condition (3.1) with $\delta \in (0, 1)$ arbitrary and $L \ge 1 - \delta$.

3) The set of fixed points of T is the entire interval $[0, 1]$. i.e., $F(T) = [0, 1]$.

Example 2. Let $T : [0, 1] \longrightarrow [0, 1]$ be given by $Tx = \frac{2}{3}$, for $x \in [0, 1)$ and $T1 = 0$. Then: 1) T satisfies (2.8) with $h \in \left[\frac{2}{3}, 1\right)$; 2) T satisfies (3.1) with $\delta \ge \frac{2}{3}$ and $L \ge \delta$; 3) T has a unique fixed point, $x^* = \frac{2}{3}$; 4) T does not satisfy (3.3).

4. Weak φ-contractions

Starting from the fact that φ-contractions are natural generalizations of strict contractions, we extended in [4] the results in [3] from weak contractions to the more general class of weak φ-contractions. To present these results, we need some concepts from Rus [30], [32] and Berinde [1].

A map $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is called *comparison function* if it satisfies:

(i_φ) φ is monotone increasing, i.e., $t_1 < t_2 \Rightarrow \varphi(t_1) \le \varphi(t_2)$;

(ii_φ) the sequence $\{\varphi^n(t)\}_{n=0}^{\infty}$ converges to zero, for all $t \in \mathbb{R}_+$, where φ^n stands for the n^{th} iterate of φ.
If φ satisfies (i_φ) and

(iii_φ) $\sum\limits_{k=0}^{\infty} \varphi^k(t)$ converges for all $t \in \mathbb{R}_+$,

then φ is said to be a (**c**)-*comparison function* [1].
It was shown in [1] that φ satisfies (iii_φ) if and only if there exist $0 < c < 1$ and a convergent series of positive terms, $\sum\limits_{n=0}^{\infty} u_n$, such that

$$\varphi^{k+1}(t) \leq c\varphi^k(t) + u_k\,, \quad \text{for all } \ t \in \mathbb{R}_+ \quad \text{and} \ \ k \geq k_0 \text{ (fixed)}.$$

It is also known that if φ is a (c) - comparison function, then the sum of the comparison series, i.e.,

$$s(t) = \sum_{k=0}^{\infty} \varphi^k(t)\,, \quad t \in \mathbb{R}_+, \tag{4.1}$$

is monotone increasing and continuous at zero, and that any (c) - comparison function is a comparison function.

A prototype for comparison functions is

$$\varphi(t) = a\,t\,, \quad t \in \mathbb{R}_+ \quad (0 \leq a < 1)$$

but, as shown by Example 3, the comparison functions need not be neither linear, nor continuous. Note however that any comparison function is continuous at zero.

Example 3. Let $\varphi_1(t) = \dfrac{t}{t+1}$, $t \in \mathbb{R}_+$ and $\varphi_2(t) = \dfrac{1}{2}t$, if $0 \leq t < 1$ and $\varphi_2(t) = t - \dfrac{1}{3}$, if $t \geq 1$.
Then φ_1 is a nonlinear comparison function, which is not a (c) - comparison function, while φ_2 is a discontinuous (c) - comparison function.

By replacing the well known strict contractiveness condition appearing in Theorem B, i.e., condition (2.1), by a more general one

$$d(Tx, Ty) \leq \varphi\big(d(x,y)\big)\,, \quad \text{for all } \ x, y \in X\,, \tag{4.2}$$

where φ is a certain comparison function, several fixed point theorems have been obtained, see for example Taskovic [34], Rus [32] and Berinde [1], and references therein. One of the first fixed point theorems of this type is due to Browder [5].

Recall that an operator T which satisfy a condition of the form (4.2) is commonly named φ - *contraction.*

Following the way in which the strict contractions were extended to φ - contractions, in [4] we extended Theorems 1 and 2 to weak φ - contractions.

Definition 2. [4] Let (X,d) be a metric space. A self operator $T : X \longrightarrow X$ is said to be a *weak φ-contraction* or *(φ, L)-weak contraction*, provided that there exist a comparison function φ and some $L \geq 0$, such that

$$d(Tx,Ty) \leq \varphi\big(d(x,y)\big) + L\,d(y,Tx)\,, \quad \text{for all } x,y \in X. \tag{4.3}$$

Remark 4. Clearly, any weak contraction is a weak φ - contraction, with $\varphi(t) = \delta t$, $t \in \mathbb{R}_+$ and $0 < \delta < 1$.
There exist weak φ - contractions which are not weak contractions with respect to the same metric, see Example 3.
Also, all φ - contractions are weak φ - contractions with $L \equiv 0$ in (4.3).

Remark 5. Similarly to the case of weak contractions, the fact that T satisfies (4.3), for all $x,y \in X$, does imply that the following dual inequality

$$d(Tx,Ty) \leq \varphi\big(d(x,y)\big) + L\,d(x,Ty)\,, \tag{4.4}$$

obtained from (4.3) by formally replacing $d(Tx,Ty)$ and $d(x,y)$ by $d(Ty,Tx)$ and $d(y,x)$, respectively, and then interchanging x and y, is also satisfied.

Consequently, in order to prove that a certain operator T is a weak φ -contraction, we must check the both inequalities (4.3) and (4.4).

The class of weak φ - contractions includes not only contractive type operators which have a unique fixed point, but also operators with more than one fixed point, see Example 1.

The main results in [4] consist of the next two theorems.

Theorem 4. *Let (X,d) be a complete metric space and $T : X \longrightarrow X$ a weak φ -contraction with φ a (c) - comparison function. Then*

1) *$F(T) = \{x \in X : Tx = x\} \neq \phi$;*

2) *For any $x_0 \in X$, the Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by $x_0 \in X$ and*

$$x_{n+1} = Tx_n\,, \quad n = 0,1,2,\ldots \tag{4.5}$$

converges to a fixed point x^* of T;

3) *The following estimate*

$$d(x_n,x^*) \leq s\big(d(x_n,x_{n+1})\big)\,, \quad n = 0,1,2,\ldots \tag{4.6}$$

holds, where $s(t)$ is given by (4.1).

Theorem 5. *Let X and T as in Theorem 4. Suppose T also satisfies the following condition: there exist a comparison function ψ and some $L_1 \geq 0$ such that*

$$d(Tx,Ty) \leq \psi\big(d(x,y)\big) + L_1 d(x,Tx)\,, \tag{4.7}$$

holds, for all $x,y \in X$.
Then

1) *T has a unique fixed point, i.e. $F(T) = \{x^*\}$;*

2) *The estimate (4.6) holds.*

A result similar to Theorem 3, but for the more general case of weak φ-contractions is given by the next theorem.

Theorem 6. *Let (X, d) be a complete metric space, $T : X \longrightarrow X$ a (φ, L)-weak contraction and $S : X \longrightarrow X$ be an approximate operator of T, i.e., an operator for which there exist $\eta > 0$ such that for all $x \in X$ we have*

$$d(Tx, Sx) \leq \eta.$$

Let φ be such that $h : \mathbb{R}_+ \longrightarrow \mathbb{R}_+, h(t) = (1-L)t - \varphi(t), t > 0$ is increasing and bijective and let x_T and x_S be fixed points of T and S, respectively. Then

$$d(x_T, x_S) \leq h^{-1}(\eta).$$

Proof. We have

$$\begin{aligned} d(x_T, x_S) = d(Tx_T, Sx_S) &\leq d(Tx_T, Tx_S) + d(Tx_S, Sx_S) \\ &\leq \varphi(d(x_T, x_S)) + Ld(x_S, Tx_T) + \eta \end{aligned}$$

which implies the desired inequality. □

In order to show how general are Theorems 1-6, in the next section we shall compare the weak contractions and weak φ-contractions, on the one hand, to several contractive type conditions, on the other hand.

5. Comparing weak contractions to other contractive type mappings

As it can be seen, Theorems 2 and 4 (as well as Theorems 1 and 3, except for the uniqueness of the fixed point) preserve all conclusions in the Banach contraction principle in its complete form given by Theorem B, but under significantly weaker contractive conditions. Indeed, the contractive conditions known in literature (see Rhoades [24] and Meszaros [18]) that involve in the right-hand size the displacements

$$d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)$$

with the nonnegative coefficients

$$a(x, y), b(x, y), c(x, y), d(x, y), e(x, y),$$

respectively, are commonly based on the restrictive assumption

$$0 < a(x, y) + b(x, y) + c(x, y) + d(x, y) + e(x, y) < 1,$$

while, our condition (3.1) do not require $\delta + L$ be less than 1, thus providing a large class of contractive type mappings.

In the following we compare weak contractions to some contractive type mappings in Rhoades' classifiaction [24].

Condition (2.1) in our paper is condition (1) in Rhoades classification; Kannan condition (2.6) is condition (4); Chatterjea condition (2.7) is condition (11); Zamfirescu's conditions $(z_1) - (z_3)$ are condition (19); while condition (2.8) of Ciric is condition (24) in Rhoades classification. We also consider condition (19") in Rhoades classification: there exists $0 < h < 1$ such that

$$d(Tx,Ty) \leq h \cdot \max\left\{d(x,y), [d(x,Tx)+d(y,Ty)]/2,\right.$$
$$\left.[d(x,Ty)+d(y,Tx)]/2\right\}, \text{for all } \ x,y \in X\,.$$

It was shown by Rhoades [24] that (19") is equivalent to (19') and also to condition (19), that is, to Zamfirescu's conditions.

It is easy to show, see also Harder and Hicks [36], that conditions (21) and (21') in Rhoades classification imply (19) for $0 < h < 1/2$. Condition (21') is the following one: there exists $0 < h < 1$ such that

$$d(Tx,Ty) \leq h \cdot \max\left\{d(x,y), d(x,Tx), d(y,Ty),\right.$$
$$\left.[d(x,Ty)+d(y,Tx)]/2\right\}, \text{for all } \ x,y \in X\,.$$

It is also known that Hardy and Rogers contractive condition [37], i.e., condition (18) in Rhoades classification, imply (19) and hence the weak contractiveness condition. The former also includes Reich's condition (7) and condition (9) in Rhoades classification. Actually, condition (7) also includes the following condition

$$d(Tx,Ty) \leq \alpha d(x,y) + \beta[d(x,Tx)+d(y,Ty)],$$

where $\alpha \geq 0, \beta \geq 0$ and $\alpha + 2\beta < 1$, see Theorem 10.1.3 [29].

We may prove that other two contractive conditions in Rhoades classification, i.e., (5) and (12), also imply (19).

They are:

(5) There exists $0 < h < 1$ such that

$$d(Tx,Ty) \leq h \cdot \max\left\{d(x,Tx), d(y,Ty)\right\}, \text{for all } \ x,y \in X\,.$$

and, respectively,

(12) There exists $0 < h < 1$ such that

$$d(Tx,Ty) \leq h \cdot \max\left\{d(x,Txy), d(x,Ty)\right\}, \text{for all } \ x,y \in X\,.$$

Conclusions. Since any mapping satisfying (19) is a weak contraction, the examples above and Proposition 1 show that our theorems 1-6 include, extend, unify and complete many important fixed point theorems in literature. These also illustrate how large the class of weak contractions is.

Moreover, Theorems 3 and 6 completed the results in [3] and [4] in what concern the data dependence of fixed points in the class of weak contractions and weak φ-contractions, respectively. Some convergence theorems for the Picard iteration for various contractive type mappings, were also deduced as corollaries of the main results in [3] and [4].

There exists other fixed point theorems which intersect to our results on weak contractions, see for example Theorem 10.1.1, parts (i)-(ii) and Theorem 10.1.5 in Rus [29], which also provide classes of weakly Picard operators. It would be therefore of interest to compare Theorem 1 in our paper to the above mentioned results.

We finally mention an other open problem. It is known, see Rhoades [27], that in spite of the fact that are not continuous on the whole space X, large classes of contractive mappings that include among others the quasi contractions, are however continuous *at the fixed point*.

It is then an open question wether or not any weak contraction is continuous at the fixed point.

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