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# New stable semi-explicit Runge-Kutta methods

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ABSTRACT. Semi-explicit Runge-Kutta methods of order 3 with four stages and order 4 with five stages are discussed, from stability point of view. Some $A$-stable and $L$-stable subclasses of such methods are highlighted.

## 1. INTRODUCTION

For initial value problem

$$y'(x) = f(x, y(x)), \quad y(x_0) = y_0, \tag{1.1}$$

where $f : [a, b] \times \mathbb{R}^m \longrightarrow \mathbb{R}^m,\ x_0 = a,\ y_0, y \in \mathbb{R}^m$, we consider the numerical solution obtained by special implicit Runge-Kutta methods, called semi-explicit or diagonally-implicit (see [2], [3]).

Many authors have investigated such methods: J.C. Butcher [2], [3], E. Hairer, C. Lubich, G, Wanner [7], [8], P.J. Houwen van der and B.P. Sommeijer [9], K. Burrage [1] etc.

In this paper we will construct some subclasses of semi-explicit Runge-Kutta schemes of order 3 with four stages and order 4 with five stages for problem (1.1). These methods will be $A$-stable or $L$-stable thus they are suitable for solving stiff problems.
Without loss of generality, the analysis will be performed when (1.1) is a scalar problem.

## 2. PRELIMINARIES

Let $x_n,\ n = 0, 1, 2, \ldots, N$ be, equal spaced points from interval $[a, b]$, with $x_0 = a$, $x_n - x_{n-1} = h$, $n = 0, 1, 2 \ldots,$ $x_N = x_0 + Nh = b$, and let $y_n$ be, the approximate value of exact solution $y(x)$ of (1.1) at the point $x_n,\ n = 0, 1, 2 \ldots, N$.

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**Definition 2.1.** A semi-explicit or diagonally-implicit Runge-Kutta method with $s$ stages for numerical solution of the problem (1.1) is defined by the relations

$$k_{i,n} = hf\left(x_n^i, y_n + \sum_{j=1}^{s} a_{ij}k_{j,n}\right), \quad i = 1, 2, \ldots, s \tag{2.1}$$

$$y_{n+1} = y_n + \sum_{i=1}^{s} b_i k_{i,n};\ n = 0, 1, 2 \ldots, \tag{2.2}$$

where $x_n^i = x_n + c_i h$, and $b_i, a_{ij}, c_i,\ i, j = 1, 2, \ldots, s$ are real parameters such that

$$a_{ij} = 0, \quad \text{for} \quad j > i \quad \text{and} \quad a_{ii} = \lambda, \quad \text{for all} \quad i, j = 1, 2, \ldots, s.$$

A semi-explicit Runge-Kutta method is usually displayed in the Butcher's array

$$\begin{array}{c|c} c & A \\ \hline & b^T \end{array} \tag{2.3}$$

where $c = (c_1, c_2, \ldots, c_s)^T,\ b^T = (b_1, b_2, \ldots, b_s)$,

$$A = \begin{pmatrix} \lambda & 0 & 0 & \cdots & 0 \\ a_{21} & \lambda & 0 & \cdots & 0 \\ a_{31} & a_{32} & \lambda & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{s1} & a_{s2} & a_{s3} & \cdots & \lambda \end{pmatrix}, \tag{2.4}$$

should satisfy

$$c = A \cdot e, \tag{2.5}$$

with $e = (1, 1, \ldots, 1)^T \in \mathbb{R}^s$.

We will focus our attention only to methods with $s = 4$ stages and $s = 5$ stages. For $s = 4$ and $s = 5$ stages the Runge-Kutta methods are generated respectively by the arrays

$$\begin{array}{c|cccc} c_1 & \lambda & 0 & 0 & 0 \\ c_2 & a_{21} & \lambda & 0 & 0 \\ c_3 & a_{31} & a_{32} & \lambda & 0 \\ c_4 & a_{41} & a_{42} & a_{43} & \lambda \\ \hline & b_1 & b_2 & b_3 & b_4 \end{array} \quad \text{and} \quad \begin{array}{c|ccccc} c_1 & \lambda & 0 & 0 & 0 & 0 \\ c_2 & a_{21} & \lambda & 0 & 0 & 0 \\ c_3 & a_{31} & a_{32} & \lambda & 0 & 0 \\ c_4 & a_{41} & a_{42} & a_{43} & \lambda & 0 \\ c_5 & a_{51} & a_{52} & a_{53} & a_{54} & \lambda \\ \hline & b_1 & b_2 & b_3 & b_4 & b_5 \end{array} \tag{2.6}$$

The stability function for such methods, depends only on parameter $\lambda$ (see [9]) and it is given by

$$R(z,\lambda)=\frac{P(z,\lambda)}{Q(z,\lambda)},\quad z \text{ complex}, \tag{2.7}$$

where

$$P(z,\lambda)=1+(1-4\lambda)z+\left(\frac{1}{2}-4\lambda+6\lambda^2\right)z^2+ \\ +\left(\frac{1}{6}-2\lambda+6\lambda^2-4\lambda^3\right)z^3+\left(\frac{1}{24}-\frac{2}{3}\lambda+3\lambda^2-4\lambda^3+\lambda^4\right)z^4, \tag{2.8}$$

$$Q(z,\lambda)=(1-\lambda z)^4\,, \tag{2.9}$$

for $s=4$, and

$$P(z,\lambda)=1+(1-5\lambda)z+\left(\frac{1}{2}-5\lambda+10\lambda^2\right)z^{\lambda}+ \\ +\left(\frac{1}{6}-\frac{5}{2}\lambda+10\lambda^2-10\lambda^3\right)z^3+\left(\frac{1}{24}-\frac{5}{6}\lambda+5\lambda^2-10\lambda^3+5\lambda^4\right)z^4+ \\ +\left(\frac{1}{120}-\frac{5}{24}\lambda+\frac{5}{3}\lambda^2-5\lambda^3+5\lambda^4-\lambda^5\right)z^5, \tag{2.10}$$

$$Q(z,\lambda)=(1-\lambda z)^5\,, \tag{2.11}$$

if the methods have $s=5$ stages.

$A$-stability of the semi-explicit Runge-Kutta methods generated by the arrays (2.6) is equivalent (see [4]) to requirements that the polinomial $Q$ should have no zeros in the left half-plane, that is $\lambda>0$, and $E$-polynomial of Nørsett, [10],

$$E(y^2,\lambda):=|Q(iy,\lambda)|^2-|P(iy,\lambda)|^2\,, \tag{2.12}$$

should satisfy

$$E(y^2,\lambda)\geq 0,\quad \forall\, y\in\mathbb{R}, \tag{2.13}$$

for some $\lambda>0$.

For $L$-stability, the stability function should satisfy the $A$-stability, requirements and more

$$\lim_{|z|\to+\infty} R(z,\lambda)=0. \tag{2.14}$$

## 3. The construction of $A$-stable and $L$-stable methods of order 3

To obtain semi-explicit methods of order 3 with $s = 4$ stages, we impose the order conditions (see [2]), together with (2.5), which, in case of semi-explicit methods, become

$$b_1 + b_2 + b_3 + b_4 = 1, \tag{3.1}$$

$$b_1c_1 + b_2c_2 + b_3c_3 + b_4c_4 = \frac{1}{2}, \tag{3.2}$$

$$b_1c_1^2 + b_2c_2^2 + b_3c_3^2 + b_4c_4^2 = \frac{1}{3}, \tag{3.3}$$

$$b_2a_{21}c_1 + b_3(a_{31}c_1 + a_{32}c_2) + b_4(a_{41}c_1 + a_{42}c_2 + a_{43}c_3) = \frac{1}{6} - \frac{\lambda}{2}, \tag{3.4}$$

$$c_1 = \lambda,\ c_2 = a_{21} + \lambda,\ c_3 = a_{31} + a_{32} + \lambda, \tag{3.5}$$

$$c_4 = a_{41} + a_{42} + a_{43} + \lambda. \tag{3.6}$$

So, we are seeking values of parameters from the tableaux (2.6), usually $0 < c_i \leq 1$, $i = 1, 2, 3, 4$, that is $0 < \lambda \leq 1$, such that the equations (3.1) - (3.6) be satisfied and also (2.13) or (2.13) and (2.14).
We can state

**Lemma 3.1.** *The solutions of the system* (3.1) − (3.6) *are depending on the free parameters* $\lambda, c_2, c_3, c_4 \in (0, 1]$*, pairwise distinct and on the free parameters* $b_1, a_{31}, a_{41}, a_{42}$*, arbitrary real numbers.*
*The solutions are given by*

$$\begin{aligned} b_2 &= A_2/6(c_3 - c_2)(c_4 - c_2),\\ b_3 &= -A_3/6(c_3 - c_2)(c_4 - c_3),\\ b_4 &= A_4/6(c_4 - c_2)(c_4 - c_3),\\ a_{43} &= c_4 - a_{41} - a_{42} - \lambda,\\ a_{32} &= c_3 - a_{31} - \lambda,\ a_{21} = c_2 - \lambda,\ c_1 = \lambda, \end{aligned} \tag{3.7}$$

*where*

$$A_2 = 2 - 3(c_3 + c_4) + 6c_3c_4 - 6b_1\left[\lambda^2 - \lambda(c_3 + c_4) + c_3c_4\right], \tag{3.8}$$

$$A_3 = 2 - 3(c_2 + c_4) + 6c_2c_4 - 6b_1\left[\lambda^2 - \lambda(c_2 + c_4) + c_2c_4\right], \tag{3.9}$$

$$A_4 = 2 - 3(c_2 + c_3) + 6c_2c_3 - 6b_1\left[\lambda^2 - \lambda(c_2 + c_3) + c_2c_3\right] \tag{3.10}$$

*and the free parameters* $\lambda, c_2, c_3, c_4, b_1, a_{31}, a_{41}, a_{42}$ *should satisfy*

$$A_2\lambda(c_2-\lambda)(c_4-c_3) - A_3\left[a_{31}\lambda + c_2(c_3-a_{31}-\lambda)\right](c_4-c_2)+$$
$$+A_4\left[a_{41}\lambda + a_{42}c_2 + c_3(c_4-a_{41}-a_{42}-\lambda)\right](c_3-c_2) =$$
$$= (1-3\lambda)(c_3-c_2)(c_4-c_2)(c_4-c_3). \tag{3.11}$$

*Proof.* By solving the equations (3.1), (3.2) and (3.3) as a linear system with respect to $b_2, b_3, b_4$ and replacing the expressions of $b_2, b_3, b_4$ into the equation (3.4), and taking into account (3.5) and (3.6), after a long computation, we obtain the conclusions (3.7) and (3.11). □

**Remark 3.2.** The formulas (3.7) provide the class of all semi-explicit Runge-Kutta methods of order 3, with four stages, depending on 8 free parameters $\lambda, c_2, c_3, c_4 \in (0,1]$ distinct each other and $b_1, a_{31}, a_{41}, a_{42} \in \mathbb{R}$.

**Example 3.3.** We can check that the array

$$\begin{array}{c|cccc}
\lambda & \lambda & 0 & 0 & 0 \\
\frac{2}{5} & \frac{2}{5}-\lambda & \lambda & 0 & 0 \\
\frac{3}{5} & \frac{8}{3}\left(\frac{3}{5}-\lambda\right) & \frac{5}{3}\left(\lambda-\frac{3}{5}\right) & \lambda & 0 \\
1 & 0 & 3(1-\lambda) & 2(\lambda-1) & \lambda \\
\hline
 & 0 & \frac{10}{9} & -\frac{5}{12} & \frac{11}{36}
\end{array} \tag{3.12}$$

represents a family of semi-explicit methods of order 3 with four stages depending on one free parameter $\lambda \in (0,1]$, belonging to the class specified in the Remark 3.2.

Now we will delimit two subsets of the class of semi-explicit Runge-Kutta methods of order 3 with four stages, namely the subclass of $A$-stable methods, and the subclass of $L$-stable methods.
We have

**Theorem 3.4.** *For every* $\lambda \in [\alpha, 1], c_2, c_3, c_4 \in (0,1]$, *distinct each other and every* $b_1, a_{31}, a_{41}, a_{42}$ *arbitrary real numbers, where* $\alpha = \frac{3+\sqrt{3}}{12} = 0.3943375673\ldots$, *the solutions* (3.7) *provide a subclass of* $A$*-stable semi-explicit Runge-Kutta methods of order* 3 *with four stages.*
*Moreover, for the value* $\lambda = \lambda^* = 0.5728160625\ldots$, *and the rest of free parameters as above, the solutions* (3.7) *provide a subclass of* $L$*-stable semi-explicit Runge-Kutta methods of order* 3 *with* $s = 4$ *stages.*

*Proof.* We seek values for $\lambda$ such that the inequality (2.13) be satisfied. If we write the polynomial $E$ in the form

$$E(y^2,\lambda) = A(\lambda)y^6 + B(\lambda)y^8, \tag{3.13}$$

where $A$ and $B$ are the polynomials in $\lambda$

$$A(\lambda) = \frac{1}{72} - \frac{1}{3}\lambda + \frac{17}{6}\lambda^2 - \frac{32}{3}\lambda^3 + 17\lambda^4 - 8\lambda^5, \tag{3.14}$$

$$B(\lambda) = -\frac{1}{576} + \frac{1}{18}\lambda - \frac{25}{36}\lambda^2 + \frac{13}{3}\lambda^3 - \frac{173}{12}\lambda^4 + \frac{76}{3}\lambda^5 - 22\lambda^6 + 8\lambda^7, \tag{3.15}$$

and then, solve numerically the equations $A(\lambda) = 0$ and $B(\lambda) = 0$, we obtain the real roots. $A(\lambda)$ has the real roots

$$\lambda_1 = 0.0939962451\ldots;\ \lambda_2 = 0.3486234981\ldots,\ \lambda_3 = 1.2805797610\ldots,$$

and $B(\lambda)$ has the real roots:

$$\lambda_1' = 0.1056624327\ldots,\ \lambda_2' = 0.1072789060\ldots\ \lambda_3' = 0.2038425244\ldots,$$

$$\lambda_4' = \frac{1}{4};\ \lambda_5' = 0.3943375673\ldots$$

The polynomials $A$ and $B$ take the factorized forms

$$A(\lambda) = -8(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)(\lambda^2 - a\lambda + b), \tag{3.16}$$

$$B(\lambda) = 8(\lambda - \lambda_1')(\lambda - \lambda_2')(\lambda - \lambda_3')(\lambda - \lambda_4')(\lambda - \lambda_5')(\lambda^2 - a'\lambda + b'), \tag{3.17}$$

where $\lambda^2 - a\lambda + b$ and $\lambda^2 - a'\lambda + b'$ have no real roots, that is, they take only positive values.
We can now solve the system of inequalities

$$A(\lambda) \geq 0, \quad B(\lambda) \geq 0, \quad 0 < \lambda \leq 1, \tag{3.18}$$

by studding the sign of $A(\lambda)$ and $B(\lambda)$.
We obtain that (3.18) are satisfied if and only if $\lambda \in [\lambda_5', 1]$.
Putting $\lambda_5' = \alpha = 0.3943375673\ldots$ it follows the first conclusion of the theorem. If we take for $\lambda$ the value $\lambda = \lambda^* = 0.5728160625\ldots$, we see that the coefficient of $z^4$ in the numerator $P$ of stability function (2.7), vanishes, so the degree of $P$ is less than the degree of $Q$ and then we have (2.14). Because $\lambda^* \in [\alpha, 1]$ we can say that the subclass of semi-explicit methods with $\lambda = \lambda^*$ are $L$-stable.
We have used the Maple 6 and Mathematica 5 packages for numerical solving of equations $A(\lambda) = 0$ and $B(\lambda) = 0$. □

**Example 3.5.** We present five example of semi-explicit Runge-Kutta methods of order 3 with four stages, the first four methods are $A$-stable and the last, $L$-stable.

$$
\begin{array}{c|cccc}
\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\
\frac{1}{5} & -\frac{3}{10} & \frac{1}{2} & 0 & 0 \\
\frac{2}{3} & -\frac{1}{9} & \frac{5}{18} & \frac{1}{2} & 0 \\
1 & -\frac{4}{5} & 1 & \frac{3}{10} & \frac{1}{2} \\
\hline
 & -\frac{22}{45} & \frac{5}{9} & \frac{9}{10} & \frac{1}{30}
\end{array} \tag{3.19}
$$

$$
\begin{array}{c|cccc}
\frac{3}{5} & \frac{3}{5} & 0 & 0 & 0 \\
\frac{2}{5} & -\frac{1}{5} & \frac{3}{5} & 0 & 0 \\
\frac{1}{5} & -\frac{4}{5} & \frac{2}{5} & \frac{3}{5} & 0 \\
1 & 0 & -\frac{2}{5} & \frac{4}{5} & \frac{3}{5} \\
\hline
 & \frac{5}{12} & 0 & \frac{5}{12} & \frac{1}{6}
\end{array} \tag{3.20}
$$

$$
\begin{array}{c|cccc}
\frac{3}{4} & \frac{3}{4} & 0 & 0 & 0 \\
\frac{1}{2} & -\frac{1}{4} & \frac{3}{4} & 0 & 0 \\
\frac{2}{5} & \frac{7}{10} & -\frac{21}{20} & \frac{3}{4} & 0 \\
\frac{2}{3} & 0 & \frac{1}{3} & -\frac{5}{12} & \frac{3}{4} \\
\hline
 & \frac{184}{63} & \frac{10}{9} & \frac{425}{504} & -\frac{31}{8}
\end{array} \tag{3.21}
$$

$$
(3.22)\qquad
\begin{array}{c|cccc}
\frac{3}{5} & \frac{3}{5} & 0 & 0 & 0 \\
\frac{1}{5} & -\frac{2}{5} & \frac{3}{5} & 0 & 0 \\
\frac{2}{5} & \frac{1}{10} & -\frac{3}{10} & \frac{3}{5} & 0 \\
\frac{4}{5} & \frac{1}{5} & \frac{3}{5} & -\frac{3}{5} & \frac{3}{5} \\
\hline
 & \frac{1}{3} & \frac{5}{9} & -\frac{1}{4} & \frac{13}{36}
\end{array}
$$

$$
(3.23)\qquad
\begin{array}{c|cccc}
0.573 & 0.573 & & & \\
0.400 & -0.173 & 0.573 & & \\
0.600 & 0.072 & -0.045 & 0.573 & \\
1.000 & 0 & 1.281 & -0.854 & 0.573 \\
\hline
 & 0 & \frac{10}{9} & -\frac{5}{12} & \frac{11}{36}
\end{array}
$$

## 4. The construction of $A$-stable and $L$-stable methods of oreder 4

Now, we are focusing on semi-explicit Runge-Kutta methods, generated by the second tableau of (2.6), having order $p = 4$ and $s = 5$ stages. The order conditions become, in this case (see [2, p.170])

$$\sum_{i=1}^{5} b_i c_i^{k-1} = \frac{1}{k}, \quad k = 1, 2, 3, 4 \tag{4.1}$$

$$\sum_{i=2}^{5} b_i \sum_{j=1}^{i-1} a_{ij} c_j = \frac{1}{6} - \frac{\lambda}{2}, \tag{4.2}$$

$$\sum_{i=2}^{5} b_i c_i \sum_{j=1}^{i-1} a_{ij} c_j = \frac{1}{8} - \frac{\lambda}{3}, \tag{4.3}$$

$$\sum_{i=2}^{5} b_i \sum_{j=1}^{i-1} a_{ij} c_j^2 = \frac{1}{12} - \frac{\lambda}{4}, \tag{4.4}$$

$$\sum_{i=3}^{5} b_i \sum_{j=2}^{i-1} a_{ij} \sum_{k=1}^{j-1} a_{jk} c_k = \frac{1}{24} - \frac{\lambda}{3} + \frac{\lambda^2}{2}, \tag{4.5}$$

with additional conditions (2.5), that is

$$(4.6) \qquad c_1 = \lambda, \quad c_2 = a_{21} + \lambda, \quad c_3 = a_{31} + a_{32} + \lambda\,,$$

$$(4.7) \qquad c_4 = a_{41} + a_{42} + a_{43} + \lambda\,,$$

$$(4.8) \qquad c_5 = a_{51} + a_{52} + a_{53} + a_{54} + \lambda\,.$$

We are seeking for values of parameters $b_i, c_i, a_{ij}, \lambda$, usually $0 < c_i \leq 1$ $i = 1, 2, 3, 4, 5$, so $0 < \lambda \leq 1$, distinct, such that the equations (4.1)-(4.8) be satisfied and also the conditions (2.13) for $A$-stability or (2.13) and (2.14) for $L$-stability satisfied.

The complete solutions of the nonlinear algebraic system (4.1)-(4.8) is not an easy problem, but every solution of this system, provides one semi-explicit (diagonally implicit) Runge-Kutta method of order 4 with $s = 5$ stages.

We will not give the complete solutions of the system (4.1)-(4.8) because it has a very complicated form.

We can state

**Theorem 4.1.** *The solution of the system* (4.1)-(4.8) *provide A-stable Runge-Kutta methods of order* 4 *with five stages if and only if*

$$\lambda \in [\alpha, \beta] \cup [\gamma, \delta],$$

*where*

$$\alpha = 0.0701257\ldots \quad , \quad \beta = 0.0726521\ldots,$$
$$\gamma = 0.2402928\ldots \quad , \quad \delta = 0.4732683\ldots.$$

*Moreover for* $\lambda = \lambda_1 = 0.07075122\ldots$ *or* $\lambda = \lambda_2 = 0.2780538\ldots$, *the corresponding solutions of the system* (4.1)-(4.8) *provide two subclasses of L-stable Runge-Kutta methods of order* 4 *with five stages.*

*Proof.* The $E$-polynomial (2.12) for methods with $s = 5$ stages can be written, using (2.10) and (2.11), as

$$(4.9) \qquad E(y^2, \lambda) = A_1(\lambda)y^6 + A_2(\lambda)y^8 + A_3(\lambda)y^{10}\,,$$

where

$$(4.10) \qquad A_1(\lambda) = -\frac{1}{360} + \frac{1}{12}\lambda - \frac{5}{6}\lambda^2 + \frac{10}{3}\lambda^3 - 5\lambda^4 + 2\lambda^5\,,$$

$$A_2(\lambda) = \frac{1}{960} - \frac{1}{24}\lambda + \frac{47}{72}\lambda^2 - \frac{31}{6}\lambda^3 + \tag{4.11}$$
$$+ \frac{265}{12}\lambda^4 - \frac{151}{3}\lambda^5 + 55\lambda^6 - 20\lambda^7\,,$$

$$A_3(\lambda) = -\frac{1}{14400} + \frac{1}{288}\lambda - \frac{41}{576}\lambda^2 + \frac{56}{9}\lambda^3 - \frac{89}{18}\lambda^4 + \tag{4.12}$$
$$+ \frac{563}{30}\lambda^5 - \frac{505}{12}\lambda^6 + \frac{160}{3}\lambda^7 - 35\lambda^8 + 10\lambda^9\,.$$

If we find numerically the real roots of the polynomials $A_1(\lambda)$, $A_2(\lambda)$, $A_3(\lambda)$, then we can solve the inequalities

$$A_1(\lambda) \geq 0, \quad A_2(\lambda) \geq 0, \quad A_3(\lambda) \geq 0, \quad 0 < \lambda \leq 1\,, \tag{4.13}$$

by studding the sign of $A_1(\lambda)$, $A_2(\lambda)$, $A_3(\lambda)$.

We obtain that (4.13) hold for $\lambda \in [\alpha, \beta] \cup [\gamma, \delta]$, where $\alpha = 0.0701257\ldots$, $\beta = 0.072652\ldots$, $\gamma = 0.2402928404\ldots$, and $\delta = 0.4732683912\ldots$, and then with (4.9)

$$E(y^2, \lambda) \geq 0, \quad y \in \mathbb{R}, \quad \lambda \in [\alpha, \beta] \cup [\gamma, \delta]\,. \tag{4.14}$$

The last condition is sufficient for $A$-stability of every method, solution of the system (4.1)-(4.8), which ensures the order $p = 4$ of the method.

If we take $\lambda = \lambda_1 = 0.075122\ldots \in [\alpha, \beta]$ and $\lambda = \lambda_2 = 0.2780538411 \in [\gamma, \delta]$, then we can check that the coefficient of $z^5$ of the polynomial $P(z, \lambda)$ from (2.10), vanishes. In this case the condition (2.14) is fulfilled, so every solution of the system (4.1)-(4.8) with $\lambda = \lambda_1$ or $\lambda = \lambda_2$ provides one $L$-stable semi-explicit Runge-Kutta method of order $p = 4$ with $s = 5$ stages.

Again, we mention that we used the Maple 6 and Mathematica 5 packages for numerical root finding for polynomials (4.10), (4.11), (4.12).

In the next we will give only two particular solutions of the system (4.1)-(4.8), i.e. two particular $A$-stable or $L$-stable methods. □

**Example 4.2.** We present two semi-explicit Runge-Kutta methods of order 4 with five stages i.e. two solutions of the system (4.1)-(4.8), the first method (4.15) is $A$-stable and the last, (4.16), $L$-stable.

It is clearly that the coefficients of the Runge-Kutta methods (3.23) and (4.16) can be given with arbitrary accuracy.

(4.15)
$$\begin{array}{c|ccccc}
\frac{2}{5} & \frac{2}{5} & & & & \\
\frac{3}{5} & \frac{1}{5} & \frac{2}{5} & & & \\
\frac{4}{5} & \frac{1}{5} & \frac{1}{5} & \frac{2}{5} & & \\
\frac{1}{5} & \frac{1}{19} & -\frac{8}{19} & \frac{16}{95} & \frac{2}{5} & \\
1 & \frac{1337}{435} & -\frac{1934}{435} & \frac{454}{261} & \frac{304}{1305} & \frac{2}{5} \\
\hline
 & -\frac{55}{72} & \frac{5}{4} & -\frac{25}{72} & \frac{95}{144} & \frac{29}{144}
\end{array}$$

(4.16)
$$\begin{array}{c|ccccc}
0.278053 & 0.278053 & & & & \\
0.200000 & -0.078053 & 0.278053 & & & \\
0.600000 & -0.340512 & 0.667068 & 0.278053 & & \\
0.800000 & 0.905887 & 0 & -0.383939 & 0.278053 & \\
1.000000 & 20.823214 & -13.830951 & -9.962345 & 3.692028 & 0.278053 \\
\hline
 & -4.221747 & 3.262906 & 4.168667 & -2.926083 & 0.916683
\end{array}$$

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