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On a set-valued integral

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ABSTRACT. We have introduced in [5] an integral for multifunctions with respect to a multimeasure, both the multifunction and the multimeasure are taking values in the family of all nonempty compact convex subsets of a real Banach algebra. In this work we establish some relations between the integral defined in [5] and the integrals of Aumann [1], Brink - Maritz [3], Brooks [4] and Martellotti - Sambucini [9]. Then we give a Fichtenholz - Kantorovič type theorem using the integral defined in [5].

1. PRELIMINARIES AND DEFINITIONS

Throughout this paper, S be a nonempty set, $\mathcal{A}$ will be an algebra of subsets of S and $(X, \|\cdot\|)$ will be a real Banach algebra (with the identity $e \neq 0$). $\mathcal{P}_{kc}(X) = \mathcal{P}_{kc}$ is the family of all nonempty compact convex subsets of X and D is the Hausdorff metric defined on $\mathcal{P}_{kc}$. We define $\|B\| = D(B, O) = \sup_{x \in B} \|x\|$ for any $B \in \mathcal{P}_{kc}$, where $O = \{0\}$.

If $A, B \subseteq X$ and $\lambda \in \mathbb{R}$, then $A \cdot B = \{xy | x \in A, y \in B\}$, $\lambda A = \{\lambda x | x \in A\}$,

$$(*) \qquad A + B = \{x + y | x \in A, y \in B\}.$$

$\mathcal{K} \subset \mathcal{P}_{kc}$ will be a semigroup with identity O under the operation (*) such that:

(K_1) $\quad A \cdot B \in \mathcal{K}$ for any $A, B \in \mathcal{K}$,

(K_2) $\quad A \cdot (B + C) = A \cdot B + A \cdot C$ for any $A, B, C \in \mathcal{K}$,

(K_3) $\quad \lambda A \in \mathcal{K}$ for any $\lambda \in \mathbb{R}_+, A \in \mathcal{K}$,

(K_4) $\quad \{e\} \in \mathcal{K}$.

Example 1.1. *If $X = \mathbb{R}$, then $\mathcal{K} = \{A | A \in \mathcal{P}_{kc}(\mathbb{R}), A \subset \mathbb{R}_+\}$ satisfies $(K_1) - (K_4)$.*

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Definition 1.1. *A multifunction $\varphi : \mathcal{A} \to \mathcal{P}_{kc}$ is said to be a multimeasure (cf. [4]) if:*

(i) $$\varphi(\phi) = O,$$

(ii) $$\varphi(E_1 \cup E_2) = \varphi(E_1) + \varphi(E_2), \forall E_1, E_2 \in \mathcal{A}, E_1 \cap E_2 = \phi.$$

Definition 1.2. *Let $\varphi : \mathcal{A} \to \mathcal{P}_{kc}$. The variation of φ is the function $\nu(\varphi, \cdot) : \mathcal{A} \to [0, +\infty]$,*

$$\nu(\varphi, A) = \sup \left\{ \sum_{i=1}^{n} \|\varphi(E_i)\|; (E_i)_{i=1}^{n} \subset \mathcal{A}, E_i \cap E_j = \phi (i \neq j), \bigcup_{i=1}^{n} E_i = A \right\},$$

$\forall A \in \mathcal{A}$.

If φ is a multimeasure, then $\nu(\varphi, \cdot)$ is finitely additive.

We extend $\nu(\varphi, \cdot)$ to $\mathcal{P}(S)$ as follows:

$$\widetilde{\nu}(\varphi, E) = \inf\{\nu(\varphi, A) | E \subset A, A \in \mathcal{A}\}, \quad \forall E \in \mathcal{P}(S).$$

Definition 1.3. *Let $\varphi : \mathcal{A} \to \mathcal{P}_{kc}$ be a multimeasure. A sequence $(f_n)_n$ of real functions defined on S converges in $\varphi-$measure to a function $f : S \to \mathbb{R}$ written $f_n \xrightarrow{\varphi} f$ if $f_n \xrightarrow{\nu(\varphi,\cdot)} f$ i.e. for any $\varepsilon > 0$,*

$$\lim_{n\to\infty} \widetilde{\nu}(\varphi, \{s \in S; |f_n(s) - f(s)| > \varepsilon\}) = 0 \tag{1.1}$$

(cf. [7]).

Definition 1.4. *Let $F, G : S \to \mathcal{P}_{kc}$ be multifunctions and let $\lambda \in \mathbb{R}$. We denote by $\lambda F : S \to \mathcal{P}_{kc}$, $F + G : S \to \mathcal{P}_{kc}$, $D(F, G) : S \to \mathbb{R}$, $\|F\| : S \to \mathbb{R}$ the functions defined as follows:*

$$\begin{aligned} (\lambda F)(s) &= \lambda F(s), \\ (F + G)(s) &= F(s) + G(s), \\ (D(F, G))(s) &= D(F(s), G(s)), \\ (\|F\|)(s) &= D(F(s), O), s \in S \end{aligned}$$

Throughout this paper $\varphi : \mathcal{A} \to \mathcal{K}$ will be a multimeasure and $\nu(\varphi, \cdot) = \nu$ will be the variation of φ such that $\nu(S) < +\infty$.

Definition 1.5. ([5]) *(i) A multifunction $F : S \to \mathcal{K}$ is said to be a simple multifunction if*

$$F = \sum_{i=1}^{n} C_i \cdot \chi_{A_i},$$

where $C_i \in \mathcal{K}$, $A_i \in \mathcal{A}, i \in \{1, ..., n\}$, $A_i \cap A_j = \emptyset$ $(i \neq j)$, $\bigcup_{i=1}^{n} A_i = S$ and χ_{A_i} is the characteristic function of A_i.

The integral of F over $E \in \mathcal{A}$ with respect to φ is:

$$\int_E F d\varphi = \sum_{i=1}^{n} C_i \cdot \varphi(A_i \cap E) \in \mathcal{K}.$$

(ii) A multifunction $F : S \to \mathcal{K}$ is said to be $\varphi-$totally measurable if there exists a sequence $(F_n)_n$ of simple multifunctions $F_n : S \to \mathcal{K}$ such that:

(1) $\quad D(F_n, F)$ *is $\nu-$measurable (cf.[7])* $\forall n \in \mathbb{N}$,

(2) $\quad D(F_n, F) \xrightarrow{\nu} 0$ *(cf.[7]).*

(iii) Let $F : S \to \mathcal{P}_{kc}$ be a $\varphi-$totally measurable multifunction. F is said to be $\varphi-$integrable on S if there exists a sequence $(F_n)_n$ of simple multifunctions $F_n : S \to \mathcal{K}$ satisfying the conditions (1), (2) of (ii) and:

(3)

$$\lim_{n,m\to\infty} \int_S D(F_n, F_m) d\nu = 0$$

(the integral is here in the sense of [7]).

Such a sequence $(F_n)_n$ will be said to be a defining sequence for F. In this case we define $\int_S F d\varphi = \lim_{n\to\infty} F_n d\varphi$ to be the integral of F over S with respect to φ.

Theorem 1.1. *([5]) Let $F, G : S \to \mathcal{K}$ be $\varphi-$integrable multifunctions. Then*

$$D\left(\int_E F d\varphi, \int_E G d\varphi\right) \le \int_E D(F, G) d\nu, \quad \forall E \in \mathcal{A}.$$

Remark ([5])

i) If $X = \mathbb{R}$, $\mathcal{K} = \{\{x\} | x \in \mathbb{R}\}$, $F = \{f\}$ (f is a function), $\varphi = \{\mu\}$ (μ is a finitely additive measure) and F is $\varphi-$integrable, then

$$\int_S F d\varphi = \left\{\int_S f d\mu\right\},$$

where $\int_S f d\mu$ is the Dunford integral ([7]).

ii) If $X = \mathbb{R}$, $\mathcal{K} = \{\{x\} | x \in \mathbb{R}\}$, $F = \{f\}$ (f is a function) and F is $\varphi-$integrable, then f is Brooks integrable with respect to φ and

$$(B) \int_S f d\varphi = \int_S F d\varphi,$$

where $(B) \int_S f d\varphi$ is the Brooks integral ([4]).

iii) If $X = \mathbb{R}$ and $\varphi = \{\mu\}$ (μ is a finitely additive measure), then we get $(MS) \int_S F d\mu$, the Martellotti - Sambucini integral of F with respect to μ ([9]).

Example 1.2. Let $X = \mathbb{R}$,$\mathcal{K} = \{A \subset \mathbb{R}_+ | A \in \mathcal{P}_{kc}\}$ and let $\varphi : \mathcal{A} \to \mathcal{K}$, $\varphi(E) = [\mu_1(E), \mu_2(E)]$ for any $E \in \mathcal{A}$ where $\mu_1, \mu_2 : \mathcal{A} \to \mathbb{R}_+$ are bounded finitely additive measures such that $\mu_1 \leq \mu_2$.

Let $F : S \to \mathcal{P}_{kc}$ be a multifunction. Then F can be written in the form $[f, g]$ where $f, g : S \to \mathbb{R}$ are functions satisfying $f \leq g$.

(a) If F is φ-integrable the integral of F with respect to φ is:

$$\int_E F d\varphi = \left[\int_E f d\mu_1, \int_E g d\mu_2\right], \forall E \in \mathcal{A}.$$

(b) Suppose there exists $(f_n)_n$ and $(g_n)_n$ defining sequences (with respect to μ_2) for f and g respectively, such that $0 \leq f_n \leq g_n, \forall n \in I\!N$. Then F is φ-integrable and:

$$\int_E F d\varphi = \left[\int_E f d\mu_1, \int_E g d\mu_2\right], \forall E \in \mathcal{A}.$$

Proposition 1.1. *Let $F, G : S \to \mathcal{K}$.*

(i) If F is φ-integrable and $\lambda \in I\!R_+$ then λF is $\varphi-$integrable and:

$$\int_E \lambda F d\varphi = \lambda \int_E F d\varphi, \forall E \in \mathcal{A}.$$

(ii) If F, G are φ-integrable such that $F + G$ is φ-totally measurable then $F + G$ is φ-integrable and:

$$\int_E (F + G)\, d\varphi = \int_E F d\varphi + \int_E G d\varphi, \forall E \in \mathcal{A}.$$

Proof. We use the fact that if $(F_n)_n$ and $(G_n)_n$ are defining sequences for F and G then $(\lambda F_n)_n, (F_n + G_n)_n$ are defining sequences for $\lambda F, F + G$ respectively. □

2. Relations with other integrals

Theorem 2.2. *Let $F : S \to \mathcal{K}$ be a φ-integrable multifunction. Then the function $\|F\|$ is Brooks-integrable with respect to φ. Moreover, if $X = \mathbb{R}$ and $\mathcal{K} = \{A \subset \mathbb{R}_+ | A \in \mathcal{P}_{kc}\}$ we have:*

$$(B) \int_E \|F\| d\varphi \subseteq \int_E F d\varphi, \forall E \in \mathcal{A}. \tag{2.2}$$

Proof. Let $(F_n)_n$ be a defining sequence for F. Then:

$$\int_S F d\varphi = \lim_{n\to\infty} \int_S F_n d\varphi. \tag{2.3}$$

It follows that $(\|F_n\|)_n$ is a defining sequence (in the sense of [4]) for $\|F\|$, so $\|F\|$ is Brooks-integrable with respect to φ and:

$$(B)\int_S \|F\| d\varphi = \lim_{n\to\infty}(B)\int_S \|F_n\| d\varphi. \tag{2.4}$$

Since F_n is simple,

$$(B)\int_S \|F_n\| d\varphi \subseteq \int_S F_n d\varphi, \forall n \in \mathbb{N}. \tag{2.5}$$

From (2.3), (2.4), (2.5) and the properties of Hausdorff metric it follows (2.2). □

Theorem 2.3. *If the multifunction $F : S \to \mathcal{K}$ is φ-integrable then $F$is Martellotti-Sambucini-integrable with respect to ν. Moreover, if $X = \mathbb{R}$ and $\mathcal{K} = \{A \subset \mathbb{R}_+ | A \in \mathcal{P}_{kc}\}$ it follows:*

$$(MS)\int_E F d\nu \subseteq \int_E F d\varphi, \forall E \in \mathcal{A}.$$

Proof. The proof is similar to that of previous theorem. □

Definition 2.6. *For a multimeasure $\varphi : \mathcal{A} \to \mathcal{K}$, we denote by $\mathcal{S}(\varphi)$ the set of all selection finitely additive measures of φ, i.e. $\mathcal{S}(\varphi) = \{\mu : \mathcal{A} \to X | \mu$ is a finitely additive measure and $\mu(E) \in \varphi(E), \forall E \in \mathcal{A}\}$.*

In the sequel, we suppose $\mathcal{S}(\varphi) \neq \phi$.

Theorem 2.4. *Let $F : S \to \mathcal{K}$ be a φ-integrable multifunction, let $\mu \in \mathcal{S}(\varphi)$ and we denote the variation of μ by $|\mu|$.Then F is Martellotti-Sambucini-integrable with respect to $|\mu|$. Moreover, if $X = \mathbb{R}$ and $\mathcal{K} = \{A \subset \mathbb{R}_+ | A \in \mathcal{P}_{kc}\}$ then we have:*

$$(MS)\int_E F d\mu \subseteq \int_E F d\varphi, \forall E \in \mathcal{A}.$$

Proof. Let $(F_n)_n$ be a defining sequence for F (cf. definition 1.6 - (iii)). Since $\|\mu(A)\| \le \|\varphi(A)\|$ and $|\mu|(E) \le \nu(E), \forall E \in \mathcal{A}$, it follows that $(F_n)_n$ is a defining sequence (in the sense of [9]) for F hence F is Martellotti-Sambucini-integrable with respect to $|\mu|$ and we have:

$$\int_E F d\varphi = \lim_{n\to\infty}\int_E F_n d\varphi, \forall E \in \mathcal{A}, \tag{2.6}$$

$$(MS)\int_E F d\mu = \lim_{n\to\infty}(MS)\int_E F_n d\varphi, \forall E \in \mathcal{A}. \tag{2.7}$$

We prove the last statement for $E = S$. Since F_n is simple,

$$(MS)\int_S F_n d\mu \subseteq \int_S F d\varphi, \forall n \in \mathbb{N}. \tag{2.8}$$

From (2.6), (2.7) and (2.8) it results:

$$(MS)\int_S F d\mu \subseteq \int_S F d\varphi$$

which completes the proof. □

Definition 2.7. *(i) For a multifunction $F : S \to \mathcal{K}$ and a finitely additive measure $\mu : \mathcal{A} \to X$, let $\mathcal{S}^1(F) = \{f \in L^1(\mu) | f(s) \in F(s)\mu - a.e. s \in S\}$.*

(ii) Let $F : S \to \mathcal{K}$ be a multifunction with $\mathcal{S}^1(F) \neq \phi$ and $\mu \in \mathcal{S}(\varphi)$. The Aumann integral of F with respect to μ ([1]) is

$$(A)\int_S F d\mu = \left\{\int_S f d\mu | f \in \mathcal{S}^1(F)\right\}.$$

The integral of a function $f : S \to X$ with respect to φ is

$$\widetilde{\int_S} f d\varphi = \left\{\int_S f d\mu | \mu \in \mathcal{S}(\varphi)\right\}.$$

The Brink - Maritz integral of F with respect to φ ([3]) is

$$(BM)\int_S F d\varphi = \left\{\int_S f d\mu | f \in \mathcal{S}^1(F), \mu \in \mathcal{S}(\varphi)\right\}.$$

Remark

$$(BM)\int_S F d\varphi = \cup_{\mu \in \mathcal{S}(\varphi)} \quad (A)\int_S F d\mu = \cup_{f \in \mathcal{S}^1(F)} \quad \widetilde{\int_S} f d\varphi.$$

Theorem 2.5. *Let $X = \mathbb{R}$ and $\mathcal{S}(\varphi) = \{\mu : \mathcal{A} \to \mathbb{R}_+ | \mu$ is a selection finitely additive measure of $\varphi\}$. If $F : S \to \mathcal{K}$ is a simple multifunction with $\mathcal{S}^1(F) \neq \phi$, then $(BM)\int_S F d\varphi \subseteq \int_S F d\varphi$.*

Proof. Let $\mu \in \mathcal{S}(\varphi)$ and $f \in \mathcal{S}^1(F)$. Let $F = \sum_{i=1}^{n} C_i \cdot 1_{A_i}$, $C_i \in \mathcal{K}, A_i \in \mathcal{A}$, $A_i \cap A_j = \phi (i \neq j)$, $\cup_{i=1}^n A_i = S$.

Since $f(s) \in F(s)\mu$-a.e. $s \in S$,

$$\int_S f d\mu = \sum_{i=1}^{n} \int_{A_i} f d\mu \in \sum_{i=1}^{n} C_i \cdot \varphi(A_i) = \int_S F d\varphi.$$

It follows $(BM)\int_S F d\varphi \subseteq \int_S F d\varphi$. □

Theorem 2.6. *Let $X=\mathbb{R}$, $F:S\to\mathcal{K}$ is a $\varphi-$integrable multifunction with $\mathcal{S}^1(F)\neq\phi$ and there exists a defining sequence $(F_n)_n$ for F such that $F(s)\subseteq F_{k_n}(s)\nu$-a.e. $s\in S$, $\forall n\in\mathbb{N}$, for a subsequence $(F_{k_n})_n$ of $(F_n)_n$. Then $(BM)\int_S Fd\varphi\subseteq\int_S Fd\varphi$.*

Proof. Since $(F_n)_n$ is a defining sequence of F, $\int_S Fd\varphi=\lim_{n\to\infty}\int_S F_n d\varphi$. Thus

$$\forall\varepsilon>0,\exists n_0\in\mathbb{N}\text{ such that } D\left(\int_S F_n d\varphi,\int_S Fd\varphi\right)<\varepsilon,\forall n\geq n_0. \tag{2.9}$$

Let $f\in\mathcal{S}^1(F)$ and $\mu\in\mathcal{S}(\varphi)$. From the hypothesis it results:

$$F(s)\subseteq F_{k_n}(s)\mu\text{-a.e.}s\in S,\forall n\in\mathbb{N}. \tag{2.10}$$

Cf. (2.10), $f(s)\in F_{k_n}(s)\mu$-a.e., $\forall n\in\mathbb{N}$. From theorem 2.7, we have:

$$\int_S fd\mu\in\int_S F_{k_n}d\varphi,\forall n\in\mathbb{N}. \tag{2.11}$$

From (2.9) and (2.11) it follows

$$0\leq d\left(\int_S fd\mu,\int_S Fd\varphi\right)\leq D\left(\int_S F_{k_n}d\varphi,\ \int_S Fd\varphi\right)\underset{n\to\infty}{\longrightarrow}0.$$

Thus,

$$d\left(\int_S fd\mu,\int_S Fd\varphi\right)=0\Rightarrow\int_S fd\mu\in\int_S Fd\varphi\Rightarrow(BM)\int_S Fd\varphi\subseteq\int_S Fd\varphi.$$

Theorem 2.7. *Let $X=\mathbb{R},\mathcal{K}=\{A\subset\mathbb{R}_+|A\in\mathcal{P}_{kc}\}$ and $\mathcal{S}(\varphi)=\{\mu:\mathcal{A}\to\mathbb{R}_+|\mu\,\text{is a selection finitely additive measure of }\varphi\}$. If $F:S\to\mathcal{K}$ is φ-integrable and $\mathcal{S}^1(F)\neq\emptyset$, then:*

$$(BM)\int_E Fd\varphi\subseteq\int_E Fd\varphi,\forall E\in\mathcal{A}. \tag{2.12}$$

Proof. Let $\mu\in\mathcal{S}(\varphi)$ and $F(s)=[f(s),g(s)],s\in S$. Cf. theorem 2.4, F is Martellotti-Sambucini integrable with respect to μ and:

$$(MS)\int_S Fd\mu\subseteq\int_S Fd\varphi. \tag{2.13}$$

Then f and g are μ-integrable functions and

$$(MS)\int_S Fd\mu=\left[\int_S fd\mu,\ \int_S gd\mu\right].$$

Let $h \in \mathcal{S}^1(F)$.

Then we have

$$\int_S f d\mu \leq \int_S h d\mu \leq \int_S g d\mu \text{ and thus}$$

$$\int_S h d\mu \in (MS) \int_S F d\mu. \tag{2.14}$$

From (2.13) and (2.14) we obtain: $\int_S h d\mu \in \int_S F d\varphi$ and the theorem is proved. $\square$

3. A Fichtenholz-Kantorovič type theorem

Definition 3.8. *Let $\varphi : \mathcal{A} \to \mathcal{P}_{kc}$ and $\mu : \mathcal{A} \to \mathbb{R}_+$. φ is said to be μ-continuous, written $\varphi \ll \mu$, if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $E \in \mathcal{A}, \mu(E) < \delta$ implies $v(\varphi, E) < \varepsilon$.*

Definition 3.9. *For any $A \subseteq S$ let $\gamma_A : S \to \mathcal{P}_{kc}, \gamma_A = \{e\} \cdot 1_A$.*

Definition 3.10. *Let $\mu : \mathcal{A} \to \mathbb{R}_+$ be a countably additive measure. We introduce the following spaces:*

$$\widetilde{L}^{\infty} = \{F : S \to \mathcal{P}_{kc} | \|F\| \in L^{\infty}(\mu)\},$$

$$\widetilde{L}(\varphi) = \{F : S \to \mathcal{P}_{kc} | F \text{ is } \varphi\text{-integrable and } F \in \widetilde{L}^{\infty}\}.$$

Let $d(F,G) = \|D(F,G)\|_{\infty}$, $\forall F, G \in \widetilde{L}(\varphi)$. Then d is a pseudometric over $\widetilde{L}(\varphi)$.

Theorem 3.8. *Let $\mu : A \to \mathbb{R}_+$ be a bounded countably additive measure such that $\varphi \ll \mu$ and we define $T : \widetilde{L}(\varphi) \to \mathcal{K}$ by $T(F) = \int_S F d\varphi, \forall F \in \widetilde{L}(\varphi)$. Then:*

(a) $T(\lambda F) = \lambda T(F), \forall \lambda \in \mathbb{R}_+, F \in \widetilde{L}(\varphi)$

(b) $T(F+G) = T(F) + T(G), \forall F, G \in \widetilde{L}(\varphi)$ with $F+G \in \widetilde{L}(\varphi)$

(c) there exists $r > 0$ such that $D(T(F), T(G)) \leq r d(F,G)$ whenever $F, G \in \widetilde{L}(\varphi)$ and $D(F,G)$ is $\mathcal{A}$-measurable.

Proof. (a) and (b) result from (a) and (b) of Proposition 1.10 respectively.

(c) Let $F, G \in \widetilde{L}(\varphi)$ such that $D(F,G)$ is $\mathcal{A}$-measurable. From theorem 1.7 it follows:

$$D(T(F), T(G)) = D\left(\int_S F d\varphi, \int_S G d\varphi\right) \leq \int_S D(F,G) d\nu \leq \nu(S) d(F,G).$$

Now we set $\nu(S) = r$ and the statement is proved. $\square$

Theorem 3.9. *Let S be a nonempty set, let $\mathcal{A}$ be a σ-algebra of subsets of S and let $\mu : \mathcal{A} \to \mathbb{R}_+$ be a bounded countably additive measure. Let $T : \widetilde{L}^\infty \to \mathcal{P}_{kc}$ be a multifunction such that:*

(i) $T(\lambda F) = \lambda T(F)$ for any $\lambda \in \mathbb{R}_+, F \in \widetilde{L}^\infty$,

(ii) $T(F + G) = T(F) + T(G)$ for any $F, G \in \widetilde{L}^\infty$ satisfying $F + G \in \widetilde{L}^\infty$,

(iii) there exists $r > 0$ such that $D(T(F), T(G)) \le r|\|F\| - \|G\||, \forall F, G \in \widetilde{L}^\infty$,

(iv) $\|T(\gamma_E)\| \le \mu(E), \forall E \in \mathcal{A}$,

(v) $F \in \widetilde{L}^\infty, F : S \to \mathcal{K} \Rightarrow T(F) \in \mathcal{K}$.

Then there exists a multimeasure $\varphi : \mathcal{A} \to \mathcal{K}$ such that $\nu(S) < +\infty$ (where ν is the variation of φ) with properties:

(a) $\varphi \ll \mu$,

(b) $T(\|F\|) = (B) \int_S \|F\| d\varphi, \forall F \in \widetilde{L}(\varphi)$.

Proof. (a) Let $\varphi : \mathcal{A} \to \mathcal{K}, \varphi(E) = T(\gamma_E), \forall E \in \mathcal{A}$. From (ii) it results that φ is a multimeasure and from (iv) we have $\|\varphi(E)\| \le \mu(E), \forall E \in \mathcal{A}$.

If $\{E_i\}_{i=1}^n \subset \mathcal{A}$ is any partition of S then $\sum_{i=1}^n \|\varphi(E_i)\| \le \sum_{i=1}^n \mu(E_i) = \mu(S)$. Since $\{E_i\}_{i=1}^n$ is arbitrary, $\nu(E) \le \mu(E) < +\infty$. Analogously it follows:

$$\nu(E) \le \mu(E), \forall E \in \mathcal{A}. \tag{3.15}$$

From (3.15) it results $\varphi \ll \mu$.

(b) Let $F \in \widetilde{L}(\varphi)$. So, $\|F\|$ is Brooks-integrable (cf. theorem 2.1). Let

$$b = esssup\|F\|, a = essinf\|F\|,$$

$A_i = \{s \in S | a + \frac{(b-a)i}{n} \le \|F(s)\| < a + \frac{(b-a)(i+1)}{n}\}, i \in \{0, 1, \ldots, n-2\}$,
$A_{n-1} = \{s \in S | a + \frac{(b-a)(n-1)}{n} \le \|F(s)\| \le b\}, n \in \mathbb{N}^*$ and let

$$f_n = \sum_{i=0}^{n-1} [a + \frac{(b-a)i}{n}] \cdot 1_{A_i}, n \in \mathbb{N}^*.$$

Then:
$|f_n - \|F\|| < \frac{b-a}{n}$ μ-a.e. for all $n \in \mathbb{N}^*$. Hence

$$T(\|F\|) = \lim_{n\to\infty} T(f_n) = \lim_{n\to\infty} T(\{f_n \cdot e\}) =$$
$$\lim_{n\to\infty} T\left(\sum_{i=0}^{n-1} [a + \frac{(b-a)i}{n}] \cdot \gamma_{A_i}\right) = \lim_{n\to\infty} \sum_{i=0}^{n-1} [a + \frac{(b-a)i}{n}] \cdot \varphi(A_i) =$$
$$\lim_{n\to\infty} (B) \int_S f_n d\varphi = \int_S \|F\| d\varphi$$

and thus the theorem is proved. $\square$

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