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Positive solutions of nonlinear functional-integral equations

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ABSTRACT. In this paper we study the conditions are required for existence of at least one positive solution of the functional-integral equation

$$u(x) = g(x) + \int_0^h k(x,s) F(u)(s) ds, \ \ x \in [0,h]$$

where $F : C[0,h] \to C[0,h]$ is an operator. Our approach to the problem is based on the Krasnoselskii's compression-expansion fixed point theorem.

1. INTRODUCTION

In this paper, we consider the nonlinear integral equation

$$u(x) = g(x) + \int_0^h k(x,s) F(u)(s) \, ds, \quad x \in [0,h] \tag{1.1}$$

where $F : C[0,h] \to C[0,h]$, $g : [0,h] \to \mathbb{R}$ and $k : [0,h] \times [0,h] \to \mathbb{R}$. In particular case, when F is the Nemitskii's operator attached to the function $f : [0,h] \times \mathbb{R} \to \mathbb{R}$, i.e. for $u \in C[0,h]$

$$F(u)(x) = N_f u(x) = f(x, u(x)), \ \ x \in [0,h],$$

then equation (1.1) became

$$u(x) = g(x) + \int_0^h k(x,s) f(s, u(s)) \, ds, \ \ x \in [0,h]. \tag{1.2}$$

The existence of positive solutions for (1.2) was studied in several papers [1, 2, 5, 7, 8, 9] and reference therein. For example, in [5] are established existence results of positive solutions for (1.2) and their applications to the boundary-value problem with integral boundary conditions. In [1], the equation (1.2) is used to studie the solutions for the two-point boundary value problem.

The idea of this paper was suggest in [7], where are presented existence results of multiple nonnegative continuous solutions of a nonlinear integral equation on both a compact interval and semi-infinite interval. Applications

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of this result to localize solutions of some boundary valued problem are presented in [2, 4].

The main result is obtained by applying the well known fixed point theorem due to Krasnoselskii [6]. Let us recall this result:

Theorem 1.1. [Krasnoselskii's compression-expansion fixed point theorem] *Let X be a Banach space, and let $K \subset X$ be a con in X. Assume that Ω_1, Ω_2 are two open subsets of X such that $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Consider the operator $T : K \cap \left(\overline{\Omega}_2 \backslash \Omega_1\right) \to K$ be completely continuous and either*

$$\|T(x)\| \leq \|x\|\,,\ x \in K \cap \Omega_1 \ \text{and}\ \|T(x)\| \geq \|x\|\,,\ x \in K \cap \Omega_2$$

or

$$\|T(x)\| \geq \|x\|\,,\ x \in K \cap \Omega_1 \ \text{and}\ \|T(x)\| \leq \|x\|\,,\ x \in K \cap \Omega_2$$

is true. Then T has a fixed point in $K \cap \left(\overline{\Omega}_2 \backslash \, \Omega_1\right)$.

2. Preliminary results

Consider that $C[0,h]$ is the Banach space of all continuous functions $u : [0,h] \to \mathbb{R}$, endowed with the norm $\|\cdot\|$, where

$$\|u\| = \sup_{x \in [0,h]} |u(x)|\,,\ \ u \in C[0,h]\,. \tag{2.3}$$

Let us suppose that the following conditions are satisfied:

(h_1) $0 \leq k_x(s) = k(x,s) \in L^1[0,h]$ for $x \in [0,h]$;
(h_2) the map $x \mapsto k_x$ is continuous from $[0,h]$ to $L^1[0,h]$;
(h_3) there exists $M \in (0,1)$, $\tilde{k} \in L^1[0,h]$ and an interval $[a,b] \subset [0,h]$ such that $k(x,s) \geq M\tilde{k}(s) \geq 0$, $x \in [a,b]$, a.e. $s \in [0,h]$;
(h_4) $k(x,s) \leq \tilde{k}(s)$, $x \in [0,h]$, a. e. $s \in [0,h]$;
(h_5) $g \in C[0,h]$ with $g(x) \geq 0$, $x \in [0,h]$ and $\min\limits_{a \leq x \leq b} g(x) \geq M\|g\|$.

By considering some arguments from [5], we obtain the following sufficient conditions for (h_3) and (h_4), introduced by D. O'Regan and M. Meehan in [7].

Lemma 2.1. *Assume that*
(CC) $k(\cdot,s) = \mathbf{k}$ *is nonnegative, concave and nondecreasing a.e.* $s \in [0,h]$, *then* (h_3) *and* (h_4) *hold.*

Proof. Let $\eta \in (0,h)$. Concavity of $\mathbf{k}$ implies

$$\frac{\mathbf{k}(h) - \mathbf{k}(0)}{h} \geq \frac{\mathbf{k}(h) - \mathbf{k}(\eta)}{h - \eta}.$$

Hence

$$\mathbf{k}(\eta) \geq \frac{\eta}{h}\,\mathbf{k}(h) + \left(1 - \frac{\eta}{h}\right)\mathbf{k}(0)\,. \tag{2.4}$$

Since $\mathbf{k}$ is nonnegative, we may have $\mathbf{k}(\eta) \geq M\mathbf{k}(h)$. Therefore,

$$\frac{\eta}{h}\, k(h,s) \leq k(\eta,s) \leq k(x,s)\,, x \in [\eta,h]\,,\ \text{a.e. } s \in [0,h]\,.$$

Hence, the hypothesis (h_3) is satisfied for $M = \dfrac{\eta}{h}$, $[a,b] = [\eta,h]$ and $\tilde{k}(s) = k(h,s)$ for any $\eta \in (0,h)$. □

Lemma 2.2. *If we assume that*
(CD) $k(\cdot,s) = \mathbf{k}$ *is nonnegative, concave and nonincreasing a.e.* $s \in [0,h]$, *then* (h_3) *and* (h_4) *hold.*

Proof. Concavity of $\mathbf{k}$ implies (2.4). Since $\mathbf{k}$ is nonnegative, we can choose $\mathbf{k}(\eta) \geq \left(1 - \dfrac{\eta}{h}\right)\mathbf{k}(0)$. Because $\mathbf{k}$ is nonincreasing, we get

$$\left(1 - \frac{\eta}{h}\right) k(0,s) \leq k(\eta,s) \leq k(x,s)\,, x \in [0,\eta]\,, \text{a.e. } s \in [0,h]\,.$$

Therefore, the hypothesis (h_3) is satisfied for $M = 1 - \dfrac{\eta}{h}$, $[a,b] = [0,\eta]$ and $\tilde{k}(s) = k(0,s)$ for any $\eta \in (0,h)$. □

Let $K : C[0,h] \to C[0,h]$ be an operator defined as follows

$$Kv(x) = g(x) + \int_0^h k(x,s)\, v(s)\, ds, \ \ x \in [0,h].$$

Then equation (1.1) is equivalent to the fixed point problem

$$Tu = u, \tag{2.5}$$

where $T : C[0,h] \to C[0,h]$ is given by $T = KF$.

Let $[a,b]$ be the interval given by the hypothesis (h_3) and

$$C_K = \left\{ u \in C[0,h]\,; u(x) \geq 0, x \in [0,h] \text{ and } \min_{x\in[a,b]} u(x) \geq M\|u\| \right\}.$$

Lemma 2.3. *If hypotheses* $(h_1) - (h_5)$ *are satisfied, then the operator* $K : C_K \to C[0,h]$ *is well defined, continuous and completely continuous.*

Proof. By conditions (h_1), (h_2) and (h_5) results that K is completely continuous.

Let $v \in C[0,h]$ be such that $v(x) \geq 0$, $x \in [0,h]$. By (h_1), (h_2), (h_4) and (h_5) we have

$$\begin{aligned} \|Kv\| &= \sup_{x\in[0,h]} \left(g(x) + \int_0^h k(x,s)\, v(s)\, ds \right) \\ &\leq |g|_0 + \int_0^h \tilde{k}(s)\, v(s)\, ds. \end{aligned} \tag{2.6}$$

By (h_1)–(h_3) and (h_5) we obtain

$$\begin{aligned}\min_{x\in[a,b]} Kv(x) &= \min_{x\in[a,b]} \left(g(x) + \int_a^b k(x,s)\, v(s)\, ds \right) \\ &\geq M \left(\|g\| + \int_0^h \tilde{k}(s)\, v(s) \right) \qquad (2.7)\end{aligned}$$

From (2.6) and (2.7) results $\min_{x\in[a,b]} Kv(x) \geq M \|Kv\|$. So, $Kv \in C_K$ for any $v \in C_K$. □

3. Main result

Theorem 3.2. *Suppose that* (h_1)-(h_5) *are satisfied and*

(H_1) *the operator* $F : C[0,h] \to C[0,h]$ *is continuous and increasing, i.e. for* $u, v \in C[0,h]$ *with* $u < v$ *implies* $F(u) < F(v)$*;*

(H_2) *there exists* $\alpha > 0$ *such that* $\dfrac{\alpha}{\|g\| + K_1 F(\alpha)} > 1$*, where*

$$K_1 = \sup_{0\leq x\leq h} \int_0^h k(x,s)\, ds > 0;$$

(H_3) *there exist* $\beta > 0$, $\alpha \neq \beta$ *and* $x_0 \in [0,h]$ *such that*

$$\frac{\beta}{g(x_0) + F(M\beta) \int_a^b k(x_0,s)\, ds} < 1;$$

hold. Then, (1.1) *has at least one nonnegative solution* $u \in C[0,h]$ *and either*

(A) $0 < \alpha \leq \|u\| \leq \beta$ *and* $u(x) \geq M\alpha$ *for* $x \in [a,b]$ *if* $\alpha < \beta$*;*

or

(B) $0 < \beta \leq \|u\| \leq \alpha$ *and* $u(x) \geq M\beta$ *for* $x \in [a,b]$ *if* $\alpha > \beta$*.*

holds.

Proof. Let Ω_1 and Ω_2 be a subsets in $C[0,h]$ given by

$$\Omega_1 = \{u \in C[0,h]\,;\|u\| < \alpha\} \text{ and } \Omega_2 = \{u \in C[0,h]\,;\|u\| < \beta\}$$

Define the operator $T : C_K \to C_K$ by

$$Tu(x) = KF(u)(x) = g(x) + \int_0^h k(x,s)\, F(u)(s)\, ds. \qquad (3.8)$$

Since F is continuous and using Lemma 2.3, we have T completely continuous. In what it follows, we will apply the Krasnoselskii's fixed point theorem for T. Therefore, we prove that

$$\|Tu\| < \|u\|\,, u \in C_K \cap \partial\Omega_1, \qquad (3.9)$$

and

$$\|Tu\| > \|u\|, u \in C_K \cap \partial\Omega_2. \tag{3.10}$$

Let $u \in C_K \cap \partial\Omega_1$, i.e. $u(x) \geq 0$, $x \in [0,h]$ and $\|u\| = \alpha$.
We have

$$\begin{aligned}
\sup_{x\in[0,h]} |Tu(x)| &= \sup_{x\in[0,h]} \left(g(x) + \int_0^h k(x,s) F(u)(s)\, ds \right) \\
&\leq \sup_{x\in[0,h]} g(x) + F(\alpha) \sup_{x\in[0,h]} \int_0^h k(x,s)\, ds \\
&\leq \|g\| + K_1 F(\alpha) \\
&< \alpha = \|u\|. \qquad (3.11)
\end{aligned}$$

If $u \in C_K \cap \partial\Omega_2$, then $u(x) \geq 0$, $x \in [0,h]$, $\|u\| = \beta$ and $M\beta \leq u(x) \leq \beta$, $x \in [a,b]$. Then

$$\begin{aligned}
T(u)(x_0) &= g(x_0) + \int_0^h k(x_0,s) F(u)(s)\, ds \\
&\geq g(x_0) + \int_a^b k(x_0,s) F(u)(s)\, ds \\
&\geq g(x_0) + F(M\beta) \int_a^b k(x_0,s)\, ds \\
&> \beta = \|u\|. \qquad (3.12)
\end{aligned}$$

Therefore, (3.11), (3.12) proves (3.9), (3.10) respectively. □

If we replace the hypothesis (h_3) and (h_4) with (CC) from Lemma 2.1 we get the following result

Corollary 3.1. *Suppose that* (h_1), (h_2), (CC), (h_5), (H_1) *and*
(HC_1) *there exists* $\alpha > 0$ *such that* $\alpha > \|g\| + F(\alpha) \int_0^h k(h,s)\, ds$;
(HC_2) *there exists* $\beta > 0$, $\beta \neq \alpha$ *such that for* $\eta \in (0,h)$ *we have*

$$\beta < \min_{x\in[0,h]} g(x) + F\left(\frac{\eta\beta}{h}\right) \int_\eta^h k(0,s)\, ds.$$

Then, (1.1) *has at least one nonnegative solution* $u \in C[0,h]$ *and either*
(AC) $0 < \alpha \leq \|u\| \leq \beta$ *and* $u(x) \geq \frac{\eta\alpha}{h}$ *for* $x \in [\eta,h]$ *if* $\alpha < \beta$;
or
(BC) $0 < \beta \leq \|u\| \leq \alpha$ *and* $u(x) \geq \frac{\eta\beta}{h}$ *for* $x \in [\eta,h]$ *if* $\alpha > \beta$.
holds.

Using Lemma 2.2 we get

Corollary 3.2. *Suppose that* (h_1), (h_2), (CD), (h_5), (H_1) *and*
(HD_1) *there exists* $\alpha > 0$ *such that* $\alpha > \|g\| + F(\alpha) \int_0^h k(0,s)\, ds$;

(HD_2) *there exists* $\beta > 0$, $\beta \neq \alpha$ *such that for* $\eta \in (0,h)$ *we have*

$$\beta < \min_{x\in[0,h]} g(x) + F\left(\frac{(h-\eta)\beta}{h}\right)\int_0^\eta k(h,s)\,ds.$$

Then, (1.1) *has at least one nonnegative solution* $u \in C[0,h]$ *and either*
(AD) $0 < \alpha \leq \|u\| \leq \beta$ *and* $u(x) \geq \frac{(h-\eta)\alpha}{h}$ *for* $x \in [0,\eta]$ *if* $\alpha < \beta$;
or
(BD) $0 < \beta \leq \|u\| \leq \alpha$ *and* $u(x) \geq \frac{(h-\eta)\beta}{h}$ *for* $x \in [0,\eta]$ *if* $\alpha > \beta$.
holds.

These two corollaries contain conditions which are similar with conditions required in [5] and are very easy to check in applications.

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