CARPATHIAN J. MATH.
19 (2003), No. 1, 73 - 80

On some bivariate Hermite type interpolation and numerical integration formulae

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ABSTRACT. In this paper we present a bivariate interpolation formula using four multiple nodes and then the corresponding numerical integration formula for a double integral extended over a rectangular domain. The cubature formula (3.1) represents an extension to two variables of the Hermite quadrature formula (1.1). At (4.1) we give an integral representation for the remainder of this cubature formula.

1. INTRODUCTION

In 1878 Charles Hermite, in his famous memoir [1], where he has presented a very general extension of the Lagrange interpolation formula, has given also a two multiple points quadrature formula

$$\int_a^b f(x)dx = \sum_{i=0}^{m} \frac{(b-a)^{i+1}}{(i+1)!} \frac{\binom{m+1}{i+1}}{\binom{m+n+2}{i+1}} \cdot f^{(i)}(a) - \sum_{j=0}^{n} \frac{(a-b)^{j+1}}{(j+1)!} \frac{\binom{n+1}{j+1}}{\binom{m+n+2}{j+1}} f^{(j)}(b) + R(f) \tag{1.1}$$

which has the degree of exactness $m+n+1$.

In 1957 D.D. Stancu [6] has obtained this quadrature formula by using the Hermite two multiple nodes interpolation formula

$$f(x) = H_{m+n+2}\left(f; \begin{matrix} a \\ m+1 \end{matrix}, \begin{matrix} b \\ n+1 \end{matrix}; x\right) + R_{m+n+2}(f;x) \tag{1.2}$$

where by using a divided difference we have

Received: 06.11.2003; In revised form: 20.11.2003
2000 *Mathematics Subject Classification.* 41A05, 65D30.
Key words and phrases. *Bivariate Hermite interpolation, numerical integration.*

$$R_{m+n+2}(f;x)=(x-a)^{m+1}(x-b)^{n+1}\begin{bmatrix} x & a & b \\ 1 \, , & m+1 \, , & n+1 \, ; f\end{bmatrix}. \tag{1.3}$$

He has used also the combinatorial identity

$$\frac{n+1}{k+3}\sum_{i=0}^{m-k}\frac{\binom{k+i}{i}}{\binom{n+k+i+2}{k+2}}=\frac{\binom{m+1}{k+1}}{\binom{m+n+2}{k+1}}$$

For the remainder of the quadrature formula of Hermite (1.1) he has found the following expression

$$R(f)=\frac{(-1)^{n+1}(b-a)^{m+n+3}}{(m+n+3)!\binom{m+n+2}{n+1}}\,f^{(m+n+2)}(\xi)\,,$$

where $f\in C^{m+n+2}[a,b]$ and $a<\xi<b$.

We can make the remark that this remainder can be written also under the following form

$$R(f)=(-1)^{n+1}\frac{(b-a)^{m+n+3}}{m+n+3}\,\frac{(m+1)!(n+1)!}{[(m+n+2)!]^2}\,f^{(m+n+2)}(\xi)\,.$$

In the particular case $m=n$ the Hermite quadrature formulae becomes

$$\int_a^b f(x)dx=\sum_{j=0}^{m}\frac{(b-a)^{j+1}}{(j+1)!}\,\frac{\binom{m+1}{j+1}}{\binom{2m+2}{j+1}}\left[f^{(j)}(a)+(-1)^j f^{(j)}(b)\right]+ \\ +\frac{(-1)^{m+1}(b-a)^{2m+3}}{(2m+3)!\binom{2m+2}{m+1}}\,f^{(2m+2)}(\xi)\,.$$

We mention that this formula was given, without any proof, in a short paper of K. Petr [4], published in 1915. In 1940 N. Obreshkov [3] tried to obtain a quadrature formula with two multiple nodes by using a different method, ignoring the result of Hermite [1].

2. A bivariate interpolation formula using four multiples nodes

Assuming that we have a bivariate function f which is defined and has continuous partial derivatives of orders $(m+n+2,p+q+2)$ on the rectangle $D=[a,b]\times[c,d]$, we can see that the bivariate extension of the Hermite interpolation formula (1.2)-(1.3) is the following

$$
\begin{aligned}
(2.1) \qquad f(x,y) &= H\left(f;\, \frac{a}{m+1}, \frac{b}{n+1}, \frac{c}{p+1}; \frac{d}{q+1}; x,y\right) + \\
&+ \left(R_{m+n+2,\, p+q+2} f\right)(x,y),
\end{aligned}
$$

where, if we use the notation $f^{(s,t)}(\alpha,\beta) = \left.\dfrac{\partial^{s+t} f(x,y)}{\partial x^s \partial y^t}\right|_{\substack{x=\alpha\\ y=\beta}}$, we can write

$$
\begin{aligned}
(2.2) \quad & H\left(f;\, \frac{a}{m+1}, \frac{b}{n+1}; \frac{c}{p+1}, \frac{d}{q+1}; x,y\right) = \\
&= \sum_{i=0}^{m}\sum_{k=0}^{p} h_{1,i}(x) g_{1,k}(y) f^{(i,k)}(a,c) + \sum_{j=0}^{n}\sum_{k=0}^{p} h_{2,j}(x) g_{1,k}(y) f^{(j,k)}(b,c) + \\
&\quad + \sum_{i=0}^{m}\sum_{r=0}^{q} h_{1,i}(x) g_{2,r}(y) f^{(i,r)}(a,d) + \sum_{j=0}^{n}\sum_{r=0}^{q} h_{2,j}(x) g_{2,r}(y) f^{(j,r)}(b,d)
\end{aligned}
$$

where

$$
(2.3) \quad
\begin{cases}
h_{1,2}(x) = \left(\dfrac{x-b}{a-b}\right)^{n+1} \dfrac{(x-a)^2}{i!} \displaystyle\sum_{\nu=0}^{m-i} \binom{n+\nu}{\nu} \left(\dfrac{x-a}{b-a}\right)^{\nu} \\
h_{2,j}(x) = \left(\dfrac{x-a}{b-a}\right)^{m+1} \dfrac{(x-b)^j}{j!} \displaystyle\sum_{\mu=0}^{n-j} \binom{m+\mu}{\mu} \left(\dfrac{x-b}{a-b}\right)^{\mu} \\
g_{1,k}(y) = \left(\dfrac{y-d}{c-d}\right)^{q+1} \dfrac{(y-c)^k}{k!} \displaystyle\sum_{\nu=0}^{p-k} \binom{q+\nu}{\nu} \left(\dfrac{y-c}{d-c}\right)^{\nu} \\
g_{2,r}(y) = \left(\dfrac{y-c}{d-c}\right)^{p+1} \dfrac{(y-d)^r}{r!} \displaystyle\sum_{\mu=0}^{q-r} \binom{p+\mu}{\mu} \left(\dfrac{y-d}{c-d}\right)^{\mu}
\end{cases}
$$

The remainder of this bidimensional interpolation formula can be expressed by means of three partial divided differences

$$(2.4)\quad \left(R_{m+n+2,\,p+q+2}f\right)(x,y)=$$

$$=(x-a)^{m+1}(x-b)^{n+1}\left[x;\begin{matrix}a\\ m+1\end{matrix},\begin{matrix}b\\ n+1\end{matrix}f(t,y)\right]_t+$$

$$+(y-c)^{p+1}(y-d)^{q+1}\left[y;\begin{matrix}c\\ p+1\end{matrix},\begin{matrix}d\\ q+1\end{matrix};f(x,z)\right]_z-$$

$$-(x-a)^{m+1}(x-b)^{n+1}(y-c)^{p+1}(y-d)^{q+1}\cdot$$

$$\cdot\left[x,y;\begin{matrix}a\\ m+1\end{matrix},\begin{matrix}b\\ n+1\end{matrix};\begin{matrix}c\\ p+1\end{matrix},\begin{matrix}d\\ q+1\end{matrix};f(t,\tau)\right].$$

If we apply the law of the mean to the divided differences occurring above we can get the following expression for this remainder

$$\left(R_{m+n+2\,,p+q+2}\ f\right)(x,y)=$$

$$=\frac{(x-a)^{m+1}(x-b)^{n+1}}{(m+n+2)!}\,f^{(m+n+2,0)}(\xi,y)+$$

$$+\frac{(y-c)^{p+1}(y-d)^{q+1}}{(p+q+2)!}\,f^{(0,p+q+2)}(x,\eta)-$$

$$-\frac{(x-a)^{m+1}(x-b)^{n+1}(y-c)^{p+1}(y-d)^{q+1}}{(m+n+2)!(p+q+2)!}\,f^{(m+n+2,\,p+q+2)}(\xi,\eta)$$

where $\xi\in(a,b)$ and $\eta\in(c,d)$.

3. Construction of the Hermite type cubature formula

Integrating over the rectangle D the bivariate interpolation formula (2.1)-(2.4) we can obtain an extension to two variables of the quadrature formula of Hermite, namely

$$(3.1)\qquad \iint_D f(x,y)dx\,dy = \sum_{i=0}^{m}\sum_{k=0}^{p} A_i^{m,n}(a,b)C_k^{p,q}(c,d)f^{(i,k)}(a,c)-$$

$$-\sum_{j=0}^{n}\sum_{k=0}^{p} B_j^{m,n}(a,b)C_k^{p,q}(c,d)f^{(j,k)}(b,c)-$$

$$-\sum_{i=0}^{m}\sum_{r=0}^{q} A_i^{m,n}(a,b)C_r^{p,q}(c,d)f^{(i,r)}(a,d)+$$

$$+\sum_{j=0}^{n}\sum_{r=0}^{q} B_j^{m,n}(a,c)C_r^{p,q}(c,d)f^{(j,r)}(b,d) + R(f)\,,$$

where

$$A_i^{m,n}(a,b) = \frac{(b-a)^{i+1}}{(i+1)!}\,\frac{\binom{m+1}{i+1}}{\binom{m+n+2}{i+1}}\,,$$

$$B_j^{(m,n)}(a,b) = \frac{(a-b)^{j+1}}{(j+1)!}\,\frac{\binom{n+1}{j+1}}{\binom{m+n+2}{j+1}}\,,$$

$$C_k^{p,q}(c,d) = \frac{(d-c)^{k+1}}{(k+1)!}\,\frac{\binom{p+1}{k+1}}{\binom{p+q+2}{k+1}}\,,$$

$$D_r^{p,q}(c,d) = \frac{(c-d)^{r+1}}{(r+1)!}\,\frac{\binom{p+1}{r+1}}{\binom{p+q+2}{r+1}}\,.$$

The remainder of this cubature formula has the following expression

$$(3.2)\quad R(f)=\frac{(-1)^{n+1}(b-a)^{m+n+3}}{(m+n+3)!\binom{m+n+2}{n+1}}\,f^{(m+n+2,\,0)}(\xi,\eta_1)+$$

$$+\frac{(-1)^{q+1}(d-c)^{p+q+3}}{(p+q+3)!\binom{p+q+2}{q+1}}\,f^{(0,\,p+q+2)}(\xi_1,\eta)+$$

$$+\frac{(-1)^{n+q+1}(b-a)^{m+n+3}(d-c)^{p+q+3}}{(m+n+3)!(p+q+3)!\binom{m+n+2}{n+1}\binom{p+q+2}{q+1}}\,f^{(m+n+2,\,p+q+2)}(\xi,\eta)\,,$$

where ξ,ξ_1 belong to the interval (a,b), while η and η_1 belong to the interval (c,d).

At (3.1)-(3.2) we have an extension to two variables of the Hermite quadrature formula (1.1).

4. An integral representation of the remainder of the cubature formula (3.1)

If we take into consideration the result of D.D. Stancu [9] concerning the expression of the remainder in some linear approximation formulae in two variables, we can give the following integral representation for the remainder of the cubature formula (3.1)

$$(4.1)\quad R(f)=\iint_D\Big[G(t)f^{(m+n+2,\,0)}(t,z)+H(z)f^{(0,\,p+q+2)}(t,z)-$$

$$-\,G(t)H(z)f^{(m+n+2,\,p+q+2)}(t,z)\Big]\,dt\,dz$$

where, if we use the factorial power, we can write the first Peano kernel in the form

$$G(t)=\frac{1}{(m+n+1)!}\,R_1\left((x-t)_+^{m+n+1}\right)=\frac{(b-t)^{m+n+1}}{(m+n+1)!}-$$

$$-\sum_{j=0}^{n}\frac{(a-b)^{j+1}}{(j+1)!}\,\frac{\binom{n+1}{j+1}}{\binom{m+n+2}{j+1}}\,(m+n+1)^{[j]}(b-t)^{m+n-j+1}\,.$$

The second Peano kernel is

$$H(z)=\frac{1}{(p+q+1)!}R_2\left((y-z)_+^{p+q+1}\right)=\frac{(d-z)^{p+q+1}}{(p+q+1)!}-$$

$$-\sum_{k=0}^{q}\frac{(c-d)^{k+1}}{(k+1)!}\frac{\binom{q+1}{k+1}}{\binom{p+q+2}{k+1}}(p+q+1)^{[k]}(d-z)^{p+q-k+1}.$$

We can write

$$\frac{1}{(j+1)!}\frac{1}{\binom{m+n+2}{j+1}}(m+n+1)^{[j]}=\frac{1}{m+n+2}$$

and

$$-(a-b)^{j+1}(b-t)^{m+n+1-j}=(b-t)^{m+n+1}\left(\frac{a-b}{b-t}\right)^j(b-a)$$

Consequently the Peano kernel $G(t)$ can be expressed by the formula

$$G(t)=\frac{(b-t)^{m+n+1}}{(m+n+1)!}+\frac{1}{m+n+2}\sum_{j=0}^{n}\binom{n+1}{j+1}\left(\frac{a-b}{b-t}\right)^j(b-t)^{m+n+1}(b-a)$$

In a similar way we finds

$$H(z)=\frac{(d-z)^{p+q+1}}{(p+q+1)!}+\frac{1}{p+q+2}\sum_{k=0}^{q}\binom{q+1}{k+1}\left(\frac{c-d}{d-z}\right)^k(d-z)^{p+q+1}(d-c).$$

By applying the law of the mean to the integrals occurring in (4.1) we can obtain the expression (3.2) for the remainder $R(f)$.

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