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Summable almost stability of fixed point iteration procedures

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ABSTRACT. A finer concept of almost stability for fixed point iteration procedures is introduced and studied. We show that Picard, Kirk's, Mann and Ishikawa iteration procedures, which are known to be almost stable and stable with respect to some classes of contractive operators, are also summably almost stable.

1. INTRODUCTION

There is a close relationship between the problem of solving a nonlinear equation and that of approximating fixed points of a corresponding contractive type operator. Consequently, there is a theoretical and practical interest in approximating fixed points of various contractive type operators. There exist several methods for approximating fixed points: Picard iteration, the most used for strict contractive type operators, Krasnoselskij iteration, very useful for approximating fixed points of weak contractive operators, like nonexpansive or pseudo-contractive type operators, Mann and Ishikawa iterations etc., see for example the recent monograph [3].

In view of their concrete applications, it is of great interest to know whether these methods are numerically *stable* or not. Roughly speaking, a fixed point iteration is numerically stable if small perturbations (due to approximation, rounding errors etc.) during computations, will produce small changes on the approximate value of the fixed point computed by means of this method.

Since the publication of the results of Harder [12], Harder and Hicks [13], [14], where a concept of stable fixed point iteration procedure was introduced and studied, many other stability results for several fixed point iteration procedures and for various classes of nonlinear operators were obtained in [29],[30],[16]-[27],[2], see also [3], Chapter 7.

All these papers use the concept of *stability* introduced by Harder [12] or a slightly weaker form, called *almost stability*, introduced by Osilike [26] and also adopted by other authors [16].

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This paper proposes a finer concept of almost stability, called *summable almost stability*, that can be appropriately adapted for the usual notion of stability of fixed point iteration procedures.

2. Stability of fixed point iteration procedures

Let (X, d) be a metric space, T a self map of X with $F_T = \{x \in X : Tx = x\} \neq \varnothing$, and consider a fixed point iteration procedure, i.e., a sequence $\{x_n\}_{n=0}^{\infty}$ defined by $x_0 \in X$ and

$$x_{n+1} = f(T, x_n), \quad n = 0, 1, 2, \dots, \tag{1}$$

where f is some function.

Examples of such iterations are: Picard iteration, obtained for $f(T, x_n) = Tx_n$; Krasnoselskij iteration, obtained for $f(T, x_n) = (1 - \lambda)x_n + \lambda Tx_n$, $\lambda \in [0, 1]$; Mann iteration, obtained for $f(T, x_n) = (1 - a_n)x_n + a_n Tx_n$, $\{a_n\}_{n=0}^{\infty} \subset [0, 1]$ and Ishikawa iteration, obtained for $f(T, x_n) = (1 - \alpha_n)x_n + \alpha_n T\big[(1 - \beta_n)x_n + \beta_n Tx_n\big]$, with $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty} \subset [0, 1]$ (the last three methods are defined in a normed linear space).

Suppose $\{x_n\}_{n=0}^{\infty}$ converges strongly to some $p \in F_T$. In concrete applications, when computing $\{x_n\}_{n=0}^{\infty}$ we usually follow the next steps:

1. Choose the initial guess (approximation) $x_0 \in X$;
2. Compute $x_1 = f(T, x_0)$. But, due to various errors that occur during calculations (rounding errors, numerical approximations of functions, derivatives, integrals etc.), we do not obtain the exact value of x_1, but a different one, say y_1, which is however close enough to x_1, i.e., $y_1 \approx x_1$;
3. Consequently, when computing $x_2 = f(T, x_1)$ we will actually obtain x_2 as

$$x_2 = f(T, y_1).$$

So, again, instead of the theoretical value of x_2 we expect, another value y_2 will be obtained, y_2 being close enough to x_2, i.e., $y_2 \approx x_2$, and so on.

In this way, instead of the theoretical sequence $\{x_n\}_{n=0}^{\infty}$ defined by the iteration (1), we obtain practically an *approximate* sequence $\{y_n\}_{n=0}^{\infty}$.

We shall consider the given fixed point iteration method to be numerically *stable* if and only if, for y_n close enough to x_n at each stage, the approximate sequence $\{y_n\}_{n=0}^{\infty}$ still converges to the fixed point p of T. The next definition, due to Harder and Hicks [14], expresses basically the previous idea.

Definition 1. Let (X, d) be a metric space, $T : X \longrightarrow X$ a self map and $x_0 \in X$. Assume that the iteration procedure (1) converges to a fixed point p of T. Let $\{y_n\}_{n=0}^{\infty}$ be an arbitrary sequence in X and set

$$\varepsilon_n = d\big(y_{n+1}, f(T, y_n)\big), \quad n = 0, 1, 2, \dots. \tag{2}$$

The fixed point procedure (1) is said to be *T-stable* or *stable with respect to T* if and only if

$$\lim_{n\to\infty} \varepsilon_n = 0 \Rightarrow \lim_{n\to\infty} y_n = p\,. \tag{3}$$

Using the notion given in Definition 1, Harder and Hicks [13], [14] verified the stability of various iteration procedures (Picard, Mann and Kirk's iterations) with respect to several classes of contractive type operators. Rhoades [29], [30] extended the results of Harder and Hicks [14] to two different classes of contractive operators of Ciric's type. Osilike [17]-[26] further investigated the stability of Mann and Ishikawa iterations for a large class of contractive type operators T that satisfy a condition of the form

$$d(Tx, Ty) \le ad(x, y) + Ld(x, Tx)\,, \tag{4}$$

for some $a \in [0, 1)$, $L \ge 0$ and for all $x, y \in D(T)$, the domain of T.

Osilike [20], [24] and [26] also investigated the stability of Ishikawa iteration with respect to some classes of pseudocontractive operators and introduced in [26] a weaker concept of stability, called *almost stability.*

Definition 2. Let (X, d) be a metric space, $T : X \longrightarrow X$ a self map and $x_0 \in X$. Assume that the iteration procedure (1) converges to a fixed point p of T. Let $\{y_n\}_{n=0}^{\infty}$ be an arbitrary sequence in X and let $\{\epsilon_n\}_{n=0}^{\infty}$ be defined by (2). The fixed point procedure (1) is said to be *almost T-stable* or *almost stable with respect to T* if and only if

$$\sum_{n=0}^{\infty} \varepsilon_n < \infty \quad \Rightarrow \quad \lim_{n\to\infty} y_n = p\,. \tag{3'}$$

Remarks.

1) It is clear that any stable iteration procedure is also almost stable but the reverse is not generally true, as shown by Osilike [24].

2) The almost stability of Mann iteration with respect to strictly hemicontractive operators was studied in [16].

This paper considers a sharper concept of almost stability and shows some almost stable fixed point iteration procedures which are also summably almost stable with respect to some classes of contractive operators.

3. Summable almost stability of fixed point iteration procedures

Definition 3. Let (X, d) be a metric space, $T : X \longrightarrow X$ a self map and $x_0 \in X$. Assume that the iteration procedure (1) converges to a fixed point p of T. Let $\{y_n\}_{n=0}^{\infty}$ be an arbitrary sequence in X and let $\{\epsilon_n\}_{n=0}^{\infty}$

be defined by (2). The fixed point procedure (1) is said to be *summably almost T-stable* or *summably almost stable with respect to T* if and only if

$$\sum_{n=0}^{\infty} \varepsilon_n < \infty \quad \Rightarrow \quad \sum_{n=0}^{\infty} d(y_n, p) < \infty. \tag{5}$$

Remarks.

1) It is obvious that any almost stable iteration procedure is also summably almost stable, since

$$\sum_{n=0}^{\infty} d(y_n, p) < \infty \Rightarrow \lim_{n\to\infty} y_n = p\,,$$

but the reverse is generally not true.

There exist fixed point iteration procedures which are not summably almost stable, as shown by Example 1;

2) The summable almost stability of a fixed point iteration procedure actually expresses a very important property regarding the rate of convergence of the sequence $\{y_n\}_{n=0}^{\infty}$, converging to the fixed point p, i.e., the fact that the "displacements" $d(y_n, p)$ converge fast enough to 0 to ensure the convergence of the series $\sum_{n=0}^{\infty} d(y_n, p)$.

3) We can introduce similarly the concept of summable stability and study the summable stability of fixed point iteration procedures which are known to be stable.

Example 1. Let $T : [0, 1] \longrightarrow [0, 1]$ be defined by $Tx = x$, for each $x \in [0, 1]$, where $[0, 1]$ has the usual metric. Then T is continuous, nonexpansive and $F_T = [0, 1]$. It is known that Picard iteration is not T-stable (and hence nor almost T-stable). We shall show that Picard iteration is not summably almost T-stable, too.

Indeed, let $p = 0$. Take $y_n = \frac{1}{n}$, for all $n \geq 1$. Then $\lim_{n\to\infty} y_n = 0$,

$$\epsilon_n = |y_{n+1} - Ty_n| \;=\; \tfrac{1}{n(n+1)} \;\text{ and hence }\; \textstyle\sum_{n=0}^{\infty} \varepsilon_n < \infty,$$

but

$$\textstyle\sum_{n=0}^{\infty} \|y_n - p\| = \sum_{n=0}^{\infty} \frac{1}{n} = \infty.$$

Therefore Picard iteration is not summably almost T-stable.

Theorems 1-3 in this paper will show some fixed point iteration procedures which are summably almost stable.

The key role in proving the main results will be played by part (ii) of Lemma 1.6 [3], which is actually the generalized ratio test for positive series, see also [1] and references therein.

Lemma 1. *Let $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ be sequences of nonnegative numbers and $0 \leq q < 1$, such that*

$$a_{n+1} \leq q a_n + b_n, \textit{for all } n \geq 0.$$

(i) If $\lim_{n\to\infty} b_n = 0$, *then* $\lim_{n\to\infty} a_n = 0$.

(ii) If

$$\sum_{n=0}^{\infty} b_n < \infty, \ \ \textit{then} \ \sum_{n=0}^{\infty} a_n < \infty.$$

The main results of this paper are given by Theorems 1-3 following.

Theorem 1. *Let* (X, d) *be a metric space and* $T : X \to X$ *a mapping satisfying (4). Suppose* T *has a fixed point* p. *Let* $x_0 \in X$ *and* $x_{n+1} = Tx_n,\ n \geq 0$.

Then $\{x_n\}$ *converges strongly to* p *and is summable almost stable with respect to* T *(i.e., for* $\{\varepsilon_n\}$ *given by (2), the implication (5) holds).*

Proof. Using the triangle rule and (4) we get

$$d(y_{n+1}, p) \leq d(y_{n+1},\ Ty_n) + d(Ty_n, p) \leq a\ d(y_n, p) + \varepsilon_n. \tag{6}$$

which by Lemma shows that Picard iteration $\{x_n\}$ is summably almost stable with respect to T.

The fact that $\{x_n\}$ converges to p is an immediate consequence of the same inequality (6). Indeed, letting $y_n \equiv x_n$ we get $\varepsilon_n = d(x_{n+1},\ Tx_n) = 0$ and so, by (6), we have

$$d(Tx_{n+1}, p) \leq ad(Tx_n, p), n \geq 0,$$

which yields

$$d(Tx_n, p) \leq a^n d(Tx_0, p) \to 0, n \to \infty,$$

that is, $\{x_n\}$ converges to p. □

Theorem 2. *Let* E *be a normed linear space and* $T : E \to E$ *a mapping satisfying (4) (with* $d(u, v) = \|a - v\|$ *). Suppose* T *has a fixed point* p. *Let* x_0 *be arbitrary in* E *and define*

$$z_n = (1 - \beta_n)x_n + \beta_n Tx_n,\ n \geq 0$$

and

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tz_n,\ n \geq 0,$$

where $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are sequences in* $[0, 1]$ *such that* $0 < \alpha \leq \alpha_n$, *for some* α.

Then $\{x_n\}$ *converges strongly to* p *and is summably almost stable with respect to* T.

Proof. Let $\{y_n\}$ be any given sequence in E and define

$$\begin{aligned} s_n &= (1 - \beta_n)y_n + \beta_n Ty_n,\ n \geq 0 \\ \varepsilon_n &= \|y_{n+1} - (1 - \alpha_n)y_n - \alpha_n Ts_n\|,\ n \geq 0. \end{aligned}$$

We have the following estimate

$$\|y_{n+1} - p\| \leq \|y_{n+1} - (1 - \alpha_n)y_n - \alpha_n Ts_n\| +$$

$$+\left\|(1-\alpha_n)(y_n-p)+\alpha_n(Ts_n-p\right\| \le$$
$$\le (1-\alpha_n)\left\|y_n-p\right\|+\alpha_n\left\|Ts_n-p\right\|+\varepsilon_n \le$$
$$\le (1-\alpha_n)\left\|y_n-p\right\|+\alpha_n a\left[(1-\beta_n)\left\|y_n-p\right\|+\beta_n a\left\|y_n-p\right\|\right]+\varepsilon_n=$$
$$=\left[(1-\alpha_n)+\alpha_n a(1-\beta_n(1-a))\right]\left\|y_n-p\right\|+\varepsilon_n \le$$
$$\le \left[1-\alpha_n(1-a)\right]\left\|y_n-p\right\|+\varepsilon_n \le \left[1-\alpha(1-a)\right]\left\|y_n-p\right\|+\varepsilon_n.$$

Since $\alpha > 0$ and $0 \le a < 1$, the conclusion follows by Lemma 1. □

Theorem 3. *Let E be a normed linear space and $T : E \to E$ a mapping satisfying (4) (with $d(u,v) = \|a-v\|$). Suppose T has a fixed point p. Let $k \ge 1$ be a fixed integer, x_0 an arbitrary point in E and define the sequence $\{x_n\}$ by*

$$x_{n+1}=\sum_{i=0}^{k}\alpha_i T^i x_n,\ n\ge 0,\ \alpha_i \ge 0,\ \alpha_1>0, \sum_{i=0}^{k}\alpha_i=1.$$

Then $\{x_n\}$ converges strongly to p and is summably almost stable with respect to T.

Proof. Let $\{y_n\}$ be any given sequence in E and define

$$\epsilon_n=\|y_{n+1}-\sum_{i=0}^{k}\alpha_i T^i y_n\|.$$

By (4), we have $\|Ty_n-p\| \le a\|y_n-p\|$. Hence

$$\|T^i y_n-p\| \le a^i\|y_n-p\|$$

and

$$\|y_{n+1}-p\| \le \left\|y_{n+1}-\sum_{i=0}^{k}\alpha_i T^i y_n\right\| + \|\sum_{i=0}^{k}\alpha_i T^i y_n - p\| \le$$
$$\le \sum_{i=0}^{k}\alpha_i\|T^i y_n-p\|+\epsilon_n \le$$
$$\le \big(\sum_{i=0}^{k}\alpha_i a^i\big)\|y_n-p\|+\epsilon_n = q|y_n-p\|+\epsilon_n,$$

where $q=\sum_{i=0}^{k}\alpha_i a^i<1$. The conclusion follows by Lemma 1. □

Remarks.

1) Theorem 1 above shows that, in a metric space, Picard iteration is summably T-stable with respect to any contractive operator T with $F_T \ne \emptyset$ which satisfies (4). Correspondingly, Theorem 2 shows that, in a normed linear space, the Ishikawa and Mann (the latter obtained for $\beta_n \equiv 0$) iteration procedures are summably T-stable with respect to the same class of operators. Theorem 3 provides a similar stability result for Kirk's fixed

point iteration procedure [16], see [14] and [25]. Theorems 1-3 in this paper correspond to Theorems 4-6 in [25], which are devoted to the study of stability of the mentioned iteration procedures.

2) It is known, see Osilike [18], [22], that condition (4) alone does not imply T has a fixed point. But if T satisfying (4) has a fixed point, it is certainly unique.

3) Many contractive type operators in Rhoades classification [28], amongst them being the strict contractions, the Kannan, Hardy and Rogers, Chatterjea and Zamfirescu operators, and partially the Ciric's quasi contractions, see for example [4], [6], do satisfy condition (4) and possess a unique fixed point. Therefore, our stability results apply to all fixed point theorems associated with these contractive mappings.

Since there are other recent (almost) stability results in literature, see for instance [16], [24], the readers are invited to find among stable (almost stable) fixed point iteration procedures those which are summably stable (almost stable) as well.

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