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Distributive Noncommutative Lattices

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Abstract. In this paper we will continue the study of distributive noncommutative lattices from [2] and [3] by considering the connection between normality and distributivity estabilished by J. Leech (Theorem 2.8,[4]) and some earlier results about distributive qvasilattices given by M.D. Gerhardts in [1].

1. Introduction

Any study about noncommutative lattices presume the raportation to the two main classes: the quasilattices and the paralattices defined in [6]. In this paper we will focus mainly on (l,l) flatt noncommutative lattices.

2. Distributive noncommutative lattices of type (G)

In [2] we have considered a class of noncommutative lattices defined as follows:

Definition 2.1. *A noncommutative lattice of type (G) is an algebraic structure $(L, \wedge, \vee)$, with two binary operations which are associative and idempotent and satisfy, for any $a, b, c \in L$:*

$$(B_1)\begin{cases} a \wedge (a \vee b) = a \\ a \vee (a \wedge b) = a \end{cases} \quad \textit{and} \quad (G)\begin{cases} (a \wedge b \wedge c) \vee (b \wedge a) = a \wedge b \\ (a \vee b \vee c) \wedge (b \vee a) = a \vee b \end{cases}$$

An algebraic structure $(B, \wedge, \vee)$ is called noncommutative rectangular lattice if $(B, \wedge)$ and $(B, \vee)$ are rectangular bands. We will say about a noncommutative lattice that splits if , it is isomorphic to a direct product of a lattice with a rectangular noncommutative lattice.Both lattices and noncommutative rectangular lattices are quasilattices and paralattices, thus, the noncommutative lattices of type (G) are simultaneously paralattices and quasilattices (see the definitions from Section 2).

Definition 2.2. *An algebraic structure $(L, \wedge, \vee)$ with two binary associative, idempotent operations is called noncommutative lattice of type (III) if it satisfy the conditions(B_1) and the following conditions:*

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$$(B_2)\begin{cases} a\wedge(b\vee a)=a\\ a\vee(b\wedge a)=a\end{cases}$$

It is known (see [7]) that this structures are (l,l) flatt quasilattices. Also it is known ([8])that any noncommutative lattice of type (G) is simultaneously of type (III). By a property estabilished in [5], the noncommutative lattices of type (G) are remarcable by the following fact:

Theorem 2.1. *A noncommutative lattice of type (III) is isomorphic to a direct product of a lattice with a rectangular noncommutative lattice if and only if it satisfies the (G) conditions.*

In a noncommutative lattice $(L,\wedge,\vee)$, we will consider the following identities:

$$\begin{array}{ll} D_l^{\vee\wedge}: & a\vee(b\wedge c)=(a\vee b)\wedge(a\vee c)\\ D_l^{\wedge\vee}: & a\wedge(b\vee c)=(a\wedge b)\vee(a\wedge c)\\ D_r^{\vee\wedge}: & (b\wedge c)\vee a=(b\wedge a)\vee(c\wedge a)\\ D_r^{\wedge\vee}: & (b\vee c)\wedge a=(b\vee a)\wedge(c\vee a)\end{array}$$

$$(D):\begin{cases} a\wedge(b\vee c)\wedge a=(a\wedge b\wedge a)\vee(a\wedge c\wedge a)\\ a\vee(b\wedge c)\vee a=(a\vee b\vee a)\wedge(a\vee c\vee a)\end{cases}$$

$$(D_b):\begin{cases} a\wedge(b\vee c)\wedge d=(a\wedge b\wedge d)\vee(a\wedge c\wedge d)\\ a\vee(b\wedge c)\vee a=(a\vee b\vee d)\wedge(a\vee c\vee d)\end{cases}$$

Definition 2.3. *a)A noncommutative lattice is called left distributive if it satisfies $D_l^{\vee\wedge}$ and $D_l^{\vee\wedge}$ and it is called right distributive if it satisfies $D_r^{\wedge\vee}$ and $D_r^{\vee\wedge}$.*

b)A noncommutative lattice is called distributive if it satisfies the D-conditions and it is called bidistributive if it satisfies the D_b-conditions.

c)A noncommutative lattice is called fully-distributive if it satisfies $D_l^{\vee\wedge}$, $D_l^{\vee\wedge}$,$D_r^{\wedge\vee}$ and $D_r^{\vee\wedge}$-conditions.

It has been prooved ([6]) that in a quasilattice the least lattice congruence is the $\mathcal{D}$ relation:

$$a\mathcal{D}b \;\Leftrightarrow\; a\wedge b\wedge a=a \;\; and \;\; b\wedge a\wedge b=b$$

for any $a,b\in L$. Thus, for any noncommutative lattice of type (G), $L/_{\mathcal{D}}$ represents the lattice factor.

Consequence 2.1. *In a noncommutative lattice of type (G),$(L,\wedge,\vee)$,*
i)$D_r^{\vee\wedge}\Leftrightarrow L/_{\mathcal{D}}$ distributive and $D_r^{\wedge\vee}\Leftrightarrow L/_{\mathcal{D}}$ distributive
ii)$D_l^{\vee\wedge}\Leftrightarrow L/_{\mathcal{D}}$ distributive and $D_l^{\vee\wedge}\Leftrightarrow L/_{\mathcal{D}}$ distributive

Proof. Let us consider a noncommutative lattice of type (G), $(L, \wedge, \vee)$, with L isomorphic to $N \times T$, N being the noncommutative rectangular factor and T being the lattice factor.

For i) it is suficient to prove that N satisfies $D_r^{\vee\wedge}$ and also satisfies the $D_r^{\wedge\vee}$ conditions, since any distributive lattice is obviously right distributive. In N we have $a \wedge b = a$ and $a \vee b = a$, for any $a, b \in N$, thanks to the (l,l) flattness. Thus, $D_r^{\wedge\vee}$ and $D_r^{\wedge\vee}$ are fullfiled. □

For ii) we notice that N satisfies also $D_l^{\vee\wedge} and D_l^{\vee\wedge}$ conditions.

Consequence 2.2. *In any noncommutative lattice of type (G), hold:*

$$L \;\; is \;\; fully-distributive \Leftrightarrow L/_{\mathcal{D}} \;\; is \;\; distributive$$
$$\Leftrightarrow D_r^{\vee\wedge} \Leftrightarrow D_r^{\wedge\vee} \Leftrightarrow D_l^{\vee\wedge} \Leftrightarrow D_l^{\wedge\vee}$$

Proof. Fully distributivity means that the four conditions

$$D_r^{\vee\wedge}, D_r^{\wedge\vee}, D_l^{\vee\wedge}, D_l^{\wedge\vee}$$

are true. By Consequence 2.1 the above result is obvious. □

Theorem 2.2. *In any noncommutative lattice of type (G) the following conditions are equivalent:*

i) For any $a, b, c \in L$,

$$a \wedge c = b \wedge c \;\; and \;\; a \vee c = b \vee c \Rightarrow a = b$$

ii) L is fully-distributive.

Proof. We have that L is isomorphic to $T \times N$, where T is a lattice and N is a noncommutative rectangular lattice which is (l,l) flatt, namely $a \wedge b = a$ and $a \vee b = a \;\forall\; a, b \in N$.

i) $\Rightarrow$ ii). We will proove that T is distibutive. Since N is allways fully distributive we will have that $T \times N$ is fully-distributive. Let us consider a_1, b_1, c_1 in T such that $a_1 \wedge c_1 = b_1 \wedge c_1$ *and* $a_1 \vee c_1 = b_1 \vee c_1$. For any a_2 in$N$ we have:

$$\begin{cases} a_1 \wedge c_1 = b_1 \wedge c_1 \\ a_1 \vee c_1 = b_1 \vee c_1 \end{cases} \Rightarrow \begin{cases} (a_1, a_2) \wedge (c_1, a_2) = (b_1, a_2) \wedge (c_1, a_2) \\ (a_1, a_2) \vee (c_1, a_2) = (b_1, a_2) \vee (c_1, a_2) \end{cases}$$

$$\Rightarrow (a_1, a_2) = (b_1, a_2) \Rightarrow a_1 = b_1$$

Thus T is distributive.

ii) $\Rightarrow$ i).If L is fully-distributive, then $L/_{\mathcal{D}}$ is distributive (from Consequence2.2), namely in $L/_{\mathcal{D}}$ the cancellation property is true. Since L is isomorphic to $L/_{\mathcal{D}} \times N$, we will consider three pairs in this product such that

$$(a_1, a_2) \wedge (c_1, c_2) = (b_1, b_2) \wedge (c_1, c_2)$$

and

$$(a_1, c_2) \vee (c_1, c_2) = (b_1, b_2) \vee (c_1, c_2).$$

But this implies that

$$\begin{cases} a_1 \wedge c_1 = b_1 \wedge c_1 \\ a_1 \vee c_1 = b_1 \vee c_1 \end{cases} \quad and \quad \begin{cases} a_2 \wedge c_2 = b_2 \wedge c_2 \\ a_2 \vee c_2 = b_2 \vee c_2 \end{cases}$$

Since $L/\mathcal{D}$ is distributive we have $a_1 = b_1$ and using the definition of operation in N we have $a_2 = b_2$.Thus,$(a_1, a_2) = (b_1, b_2)$ and in $T \times N$ the right hand cancellation rule is true. □

Theorem2.2 was given with another proof in [3]. Also there we have estabilished that the left-hand cancellation rulle implies that the noncommutative lattice of type (G) is, in this case an usual lattice.

3. Noncommutative Distributive Lattices of type (III)

Recall from [6] that an algebraic structure $(L, \wedge, \vee)$ with two binary associative, idempotent operations is called paralattice if for any a,b $\in L$:

$$B_5 : \begin{cases} a \wedge (a \vee b \vee a) = (a \vee b \vee a) \wedge a = a \\ a \vee (a \wedge b \wedge a) = (a \wedge b \wedge a) \vee a = a \end{cases}$$

and it is called quasilattice if for any $a, b \in L$:

$$B_6 : \begin{cases} a \wedge (b \vee a \vee b) \wedge a = a \\ a \vee (b \wedge a \wedge b) \vee a = a \end{cases}$$

Also, a noncommutative lattice is called (l,l) flatt if, for any $a, b \in L$:

$$(l, l) : \begin{cases} a \wedge b \wedge a = a \\ a \vee b \vee a = a \end{cases}$$

In the same way we can define the (l,r) flatt noncommutative lattice, having the restrictive semigroups $(L, \wedge)$ and $(L, \vee)$ left regular and right regular respectively. Analogously we can define the (r,l) flatt and (r,r) flatt noncommutative lattices.

In [7] was estabilished that any noncommutative lattice of type (III) is a (l,l) flatt noncommutative lattice and it is obvious then,that it is a quasilattice.

In [1] M.D. Gerhardt proved the (l,r) variant of the following result:

Theorem 3.1. *A noncommutative lattice of type (III) is isomorphic to a direct product of a distributive lattice with a (l,l) flatt rectangular noncommutative lattice.*

This result has recently been generalized in [6].First we have:

Theorem 3.2. *A distributive noncommutative lattice is a paralattice if and only if it is a quasilattice.*

The generalization of Gerhardts theorem is:

Theorem 3.3. *A bidistributive quasilattice (paralattice) factors into the direct product of a distributive lattice and a rectangular quasilattice. Conversely, every such product is a bidistributive quasilattice.*

Indeed, the noncommutative lattices of type (III) are quasilattices and the fully-distributivity obviously implies the bidistributivity.

From the definition of (l,l) flatt paralattices, they also can be defined by (B_1) and:

$$(B_3)\begin{cases}(a \vee b) \wedge a = a\\(a \vee b) \wedge a = a\end{cases}$$

Applying Theorem3.2 to (l,l) flatt lattices, we can say that , a noncommutative lattice of type (III) left-distributive is always (l,l) flatt paralattice and a left-distributive (l,l) flatt paralattice is always a noncommutative lattice of type(III).

We notice that both (l,l) flatt paralattices and noncommutative lattices of type (III) satisfy (B_1) and (l,l), namely are noncommutative lattices of type (II)([7]). From Theorem3.2 and from Theorem3.3 follows immediately:

Consequence 3.1. *In any noncommutative fully-distributive lattice of type (II), the following conditions are equivalent:*

i) L is quasilattice

ii) L is paralattice

iii) L factors into the direct product of a distributive lattice and a rectangular noncommutative lattice.

If we know what the distributivity and the bidistributivity produce on a noncommutative lattice of type (III) let us see which is the effect of the right-distributivity.

Theorem 3.4. *A noncommutative lattice of type (III) right distributive is always a paralattice.*

Proof. We have to proove that $(a \vee b) \wedge a = a$ *and* $(a \vee b) \wedge a = a \ \forall \ \ a, b \in L$.

$$(a \wedge b) \vee a = (a \vee a) \wedge (b \vee a) = a \wedge (b \vee a) = a$$

Analogously we have the the dual identity . □

Remark 3.1. *In the final of this section, let us notice that there are weaker conditions than distributivity which are sufficient for a noncommutative lattice of type (III) to be paralattice. Indeed, if a noncommutative lattice of type (III) is symmetric, namely* $a \wedge b = b \wedge a \Leftrightarrow a \vee b = b \vee a, \forall \ \ a, b \in L$, *we have that it is a paralattice. For any* $a, b \in L, a \wedge (a \wedge b) = (a \wedge b) \wedge a$ *implies* $a \vee (a \wedge b) = (a \wedge b) \vee a$. *This, together with* $a \vee (a \wedge b) = a$ *implies that* $(a \wedge b) \vee a = a$. *Analogously we have the dual identity.*

4. Distributive Binormal Noncommutative Lattices

A noncommutative lattice $(L, \wedge, \vee)$ is called binormall if, for any $a, b, c \in L$:

$$N : \begin{cases} a \wedge (b \wedge c) = a \wedge (c \wedge b) \\ a \vee (b \vee c) = a \vee (c \vee b) \end{cases}$$

From [6] we have:

Proposition 4.1. *The split quasilattices are the subvariety of fine quasilattices that are binormal. By a fine quasilattice we understand a quasilattice which is also a paralattice.*

In [9] there are defined the noncommutative lattices of type (V) as follows:

Definition 4.1. *An algebraic structure* $(L, \wedge, \vee)$ *with two binary operation which are associative and idempotent, is called noncommutatine lattice of type (V) if for any* $a, b, c \in L$ *it satisfies* (B_1) *and :*

$$and \quad N' : \begin{cases} a \wedge (b \wedge c) = a \wedge (c \wedge b) \\ a \vee (b \vee c) = a \vee (c \vee b) \end{cases}$$

Also in [9] it has been estabilished that any noncommutative lattice of type (V) is of type (III), and of type (IV), namely is a (l,l) flatt quasilattice and a binormal noncommutative lattice. We have thus, by Proposition4.1 that any noncommutative lattice of type (V) which satisfy B_3, splits. If it is bidistributive, we have by Theorem3.3 that it is a direct product of a distributive lattice and a rectangular noncommutative lattice.

In the case of skew lattices we have another interesting conection between normality and distributivity. Let us consider the axioms:

$$(B_4) \begin{cases} (b \vee a) \wedge a = a \\ (b \vee a) \wedge a = a \end{cases}$$

The noncommutative lattices $(L, \wedge, \vee)$ defined by the axioms $(B_1), (B_4)$ are called skew lattices.The skew lattices which satisfy the first axiom from N, are called normal skew lattices. These structures are interesting by the fact that they have a distributive factor, presuming just that they are symmetric([4],Th.2.8):

Theorem 4.1. *Let S be a symmetric normal skew lattice, let T be its maximal lattice image, let S^r be its reduced algebra, and let D be the maximal lattice image of S^r. The canonical epimorphisms, $S \to T$ and $S \to S^r$ induce an isomorphism of S with the fibered product over D of T with S^r, moreover, the reduced algebra S^r is distributive.*

The following theorem gives suplimentar conditions for the left distributivity to imply the conditions N'.

Definition 4.2. *In the algebraic structure $(L, \wedge, \vee)$ having the semigroups $(L, \wedge)$ and $(L, \vee)$ left regular, the subset $I \subseteq L$ is called ideal, if for any $a, b \in L$:*
$a \in I, b \in L \Rightarrow a \wedge b \in I$ *and* $a \vee b \in I$.

Let us consider an algebraic structure as in the Definition4.2

Theorem 4.2. *If in L hold the conditions $D_l^{\wedge\vee}, D_l^{\vee\wedge}$ and L is direct union of commutative ideals, then in L the N' conditions are true.*

Proof. Let us consider three elements $a, b, c \in L$.Then,

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) = (a \wedge c) \vee (a \wedge b) = a \wedge (c \vee b)$$

Here $a \wedge b$ *and* $a \wedge c$ commute since they belong to the same commutative ideal, in fact to the ideal which contains a. Analogously we have the dual condition from N'. □

Remark 4.1. *If an algebraic structure as in the Theorem4.2 is a direct union of commutative ideals, then it is symmetric. Indeed, if $x, y \in L$ are such that $x \wedge y = y \wedge x$, we have $I_x = I_y$, because $x \wedge y \in I_x$ and $y \wedge x \in I_y$. We have denoted by I_x the ideal which contains x and by I_y the ideal which contains y. Thus, x and y commute also with respect to the operation "$\vee$" since they belong to the same commutative ideal. We can ask ourself, if the algebraic structure considered in Definition4.2 is symmetric and distributive, under which hypothesis, the conditions N' or N are true.*

References

[1] Gerhardts, M.D., *Zur Characterisierung Distributiver Schiefverbande*, Math. Ann., 161(1965), 231-240

[2] Laslo, G., Fărcaş, Gh., *Distributive and Noncommutative Lattices of Type (G)*, Mathematica, Tome 38(61) (1996), nr.1-2, 105-110

[3] Laslo, G,, Fărcaş, Gh.,*On Distributive and Noncommutative Lattices of Type (G)*, Mathematica, Tome 39(62), nr. 2(1997), 239-244

[4] Leech, J.E., *Normal Skew Lattices*, Semigroup Forum, Vol. 44(1992), 1-8

[5] Gerhardts, M.D., *Uber die Zerlegbarkeit von nichtkommutativen Verbanden in kommutative Teilverbande*, Proc. Japan Acad., 41(1965), 883-888

[6] Laslo, G. and Leech, J.E., "Green's Equivalences on Noncommutative Lattices", *Acta Sci. Math.(Szeged)*, 68(2002), 501-533

[7] Farcas, Gh., *Sisteme de axiome pentru latici oblice*, St. Cerc. Mat., Tome 24, nr. 7(1972), 1989-1995

[8] Farcas, Gh., *Sur les Treillis non-commutatifs de type(G)*, Mathematica, Tome 32(55), nr.1 (1990), 15-18

[9] Farcas, Gh., *Treillis non-commutatifs de type (S)*, Studia Univ. Babeş Bolyai, Math.Tome XXXII, 2(1987), 44-48.

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