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On the p-defects of character degrees of finite groups of Lie type

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ABSTRACT. This paper is concerned with the representation theory of finite groups. According to Robinson, the truth of certain variants of Alperin's weight conjecture on the p-blocks of a finite group would imply some arithmetical conditions on the degrees of the irreducible (complex) characters of that group. The purpose of this note is to prove directly that one of these arithmetical conditions is true in the case where we consider a finite group of Lie type in good characteristic.

According to Robinson [8, Theorem 5.1] (see also [9, §5]), the truth of certain variants of Alperin's weight conjecture on the p-blocks of a finite group would imply some arithmetical conditions on the p-parts of the degrees of the irreducible (complex) characters of that group. The purpose of this note is to prove directly that one of these arithmetical conditions is true in the case where we consider a finite group of Lie type in good characteristic. (See Example 2 for the problems which arise in bad characteristic.)

Let G be a connected reductive group defined over the finite field $\mathbb{F}_q$, where q is a power of some prime p. Let $F\colon G \to G$ be the corresponding Frobenius map and G^F the finite group of fixed points. Recall that p is good for G if p is good for each simple factor involved in G, and that the conditions for the various simple types are as follows.

$$\begin{aligned} A_n:&\quad \text{no condition},\\ B_n, C_n, D_n:&\quad p \neq 2,\\ G_2, F_4, E_6, E_7:&\quad p \neq 2, 3,\\ E_8:&\quad p \neq 2, 3, 5. \end{aligned}$$

For the basic properties of finite groups of Lie type, see [1]. Now we can state:

Theorem 1. *Assume that p is a good prime for G. Let χ be an irreducible character of G^F. Then there exists an F-stable parabolic subgroup $P \subseteq G$ and an irreducible character ψ of U_P^F (where U_P is the unipotent radical of P) such that $|U_P^F|/\psi(1)$ equals the p-part of $|G^F|/\chi(1)$.*

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In order to prove this result, we first reduce to the case that G has a connected center. This can be done using regular embeddings (see [6]), as follows. We can embed G as a closed subgroup into some connected reductive group G' with a connected center, such that G' has an $\mathbb{F}_q$-rational structure compatible with that on G and G, G' have the same derived subgroup. Then we have in fact ${G'}^F = G^F {T'}^F$ for any F-stable maximal torus $T' \subseteq G'$. In particular, the quotient ${G'}^F/G^F$ is an abelian p'-group. Now, if we take any irreducible character χ' of ${G'}^F$ then, by Clifford's Theorem, the restriction of χ' is of the form $e(\chi_1 + \cdots + \chi_r)$ where $e \geq 1$ and $\chi_1, \ldots, \chi_r$ are irreducible characters of G^F which are conjugate under the action of ${G'}^F$; moreover, e divides $[{G'}^F : G^F]$. (In fact, Lusztig [6] has shown that $e = 1$ but we do not need this highly non-trivial fact here.) It follows that both e and r are prime to p, and so the terms $|{G'}^F|/\chi'(1)$ and $|G^F|/\chi_i(1)$ (where $1 \leq i \leq r$) have the same p-part. On the other hand, the collection of unipotent radicals of F-stable parabolic subgroups is the same in G and in G'. Thus, if the above theorem holds for G', then it also holds for G.

Now let us assume that the center of G is connected, and let χ be an irreducible character of G^F. We claim that there exists an F-stable unipotent class C of G such that

$$q^{\dim {}_u} \quad \text{is the } p\text{-part of } \chi(1),$$

where $u \in C$ and ${}_u$ denotes the variety of Borel subgroups containing u. This class C can be characterized in two different ways. One could either use the results in [5, Chap. 13] which describe a map ξ_G from irreducible characters to unipotent classes in terms of the Springer correspondence. Then $C = \xi_G(\pm D_G(\chi))$ where D_G denotes Alvis–Curtis–Kawanaka duality, and it remains to use the formula for $\chi(1)$ in [5, (4.26.3)]. Or one uses the fact that every irreducible character has a unipotent support (see [7] for large p, and [2] for the extension to good p). Note that here the assumption that p is good is used, since then the denominator in the formula for $\chi(1)$ in [5, (4.26.3)] is not divisible by p, which is definitely not the case in general.

Using the theory of weighted Dynkin diagrams, one can associate with an F-stable unipotent class C two subgroups $U_2, U_1 \subseteq G$ such that

(a) U_1 is the unipotent radical of some F-stable parabolic subgroup $P \subseteq G$,
(b) U_2 is a closed F-stable subgroup of U_1 which is normal in P,
(c) There exists some $u \in C \cap U_2$ such that the P-orbit of u is dense in U_2 and $C_G(u) \subseteq P$.

If p is large, this can be found in [1, Chap. 5], for example. It has been checked by Kawanaka [4, Theorem 2.1.1] that these results remain valid whenever p is a grood prime. Now let $u \in C \cap U_2$ be as in (c). Then, by

[4, Lemma 2.1.4], we have

$$\dim C_G(u) = \dim C_P(u) = \dim P - \dim U_2.$$

On the other hand, we always have

$$\dim P + \dim U_1 = \dim G = 2N + \operatorname{rank} G,$$

where N denotes the number of positive roots in G. Using also the formula $\dim C_G(u) = \operatorname{rank} G + 2 \dim_u$ (see [10, Cor. 6.5]), we find that

$$2(N - \dim_u) = \dim U_1 + \dim U_2.$$

Now Kawanaka [4, §3.1] has shown that there exists a linear character λ of U_2^F which is invariant under U_1^F and such that $U_1^F/\ker\lambda$ is an extraspecial p-group. By the character theory of extraspecial p-groups, $\dim U_1 - \dim U_2$ is even and there exists an irreducible character ψ of U_1^F which lies above λ and has degree $q^{(\dim U_1 - \dim U_2)/2}$. (Here, we use that if U is any connected unipotent group defined over $\mathbb{F}_q$, then the fixed point set under the corresponding Frobenius map has order $q^{\dim U}$.) Using the above formulas this yields that

$$\frac{|U_1^F|}{\psi(1)} = q^{\dim U_1 - (\dim U_1 - \dim U_2)/2} = q^{(\dim U_1 + \dim U_2)/2} = q^{N - \dim_u}.$$

Since q^N is exactly the p-part in $|G^F|$, the proof of Theorem 1 is complete.

Example 1. Let $G^F = \mathrm{GL}_3(q)$ and χ be of degree $q(q+1)$. The p-part of $|G^F|/\chi(1)$ is q^2. The unipotent class C associated with χ has Jordan blocks of sizes $1, 2$. The group U_1 is the group of all upper triangular unipotent matrices, U_2 is the center of U_1, and for each nontrivial linear character of U_2^F there is a unique irreducible character of U_1^F lying above it. Any such irreducible character of U_1^F has degree q.

Example 2. Let $G^F = \mathrm{Sp}_4(q)$ and χ be of degree $\frac{1}{2}q(q^2+1)$. Then the unipotent support of χ is the unique unipotent class C with $\dim_u = 1$. (This holds independently of whether q is a power of a good prime or not; see [3].) In good characteristic, the subgroups U_1, U_2 associated with C are such that $U_1 = U_2$ and $|U_1^F| = q^3$. Analogous subgroups will also exist in characteristic 2, but we cannot work with them since then the 2-defect of χ is $2q^3$. — This illustrates some of the difficulty of extending Theorem 1 to the case of bad characteristic.

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