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A class of bounded functions

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ABSTRACT. In this note we show that the Libera generalized integral operator transforms the class of convex functions and the class of functions subordinate to convex functions into the class of bounded functions.

1. INTRODUCTION

Let f be analytic in the unit disc $U = \{z : |z| < 1\}$ and let $H(U)$ denote the set of all analytic functions in U. In 1965 R. Libera showed that the operator

$$L : H(U) \to H(U)$$

defined by

$$L(f)(z) = \frac{2}{z}\int_0^z f(t)dt \tag{1.1}$$

maps S^* into S^*, where S^* is the class of starlike functions. In 1969, S. Bernardi considered the more general operator

$$L_\gamma(f)(z) = \frac{\gamma+1}{z^\gamma}\int_0^z f(t)\, t^{\gamma-1}dt \tag{1.2}$$

and showed that $L_\gamma(S^*) \subset S^*$, if $\gamma = 1, 2, \ldots$. Many authors have studied this operator where f belongs to some special class of functions.

In this paper we show that the integral operator L_γ maps K into the class of bounded functions where K denotes, as usual, the class of convex functions. Also, if f is subordinate to a convex function then $L_\gamma(f)$ is bounded too.

2. PRELIMINARIES

Let f and g be analytic in the unit disc U. We say that f is subordinate to g, written $f(z) \prec g(z)$ or $f \prec g$, if there exists a function $w(z)$ analytic in U which satisfies $w(0) = 0$, $|w(z)| < 1$ and $f(z) = g(w(z))$. If $g(z)$ is univalent in U, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.

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Let $E \subset H(U)$ and let I be an integral operator, $I : E \to H(U)$. We call I integral operator preserving subordination if

$$f \prec g \Rightarrow I[f] \prec I[g].$$

For $f \in H(U)$ and $z = re^{i\theta}$, we set

$$M_p(r, f) = \begin{cases} \left(\frac{1}{2\pi} \int_0^{2\pi} \left| f(re^{i\theta}) \right|^p d\theta \right)^{\frac{1}{p}}, \text{ for } 0 < p < \infty, \\ \sup\limits_{0 \le \theta \le 2\pi} \left| f(re^{i\theta}) \right|, \text{ for } p = \infty. \end{cases}$$

A function is said to be of Hardy spaces H^p $(0 < p < \infty)$ if $M_p(r, f)$ remains bounded as $r \nearrow 1$. H^∞ is the class of bounded analytic functions in the unit disc. We will make use of the following lemmas:

Lemma 2.1. [2] *If $f \in K$ is not of the form*

$$f(z) = a + \frac{b}{1 - ze^{i\tau}},$$

for some complex a, b and real τ, then there exists $\varepsilon = \varepsilon(f) > 0$ such that

$$f \in H^{1+\varepsilon}.$$

Lemma 2.2. (Integral theorem of Hardy-Littlewood). [1] *If $f \in H^p$ and $F(z) = \int_0^z f(t)dt$ then $F \in H^{\frac{p}{1-p}}$ for $0 < p < 1$ and $F \in H^\infty$ for $p \ge 1$.*

3. Main results

Theorem 3.1. *Let $f, g \in H(U)$, g convex, g is not of the form*

$$g(z) = a + \frac{b}{1 - ze^{i\tau}}, \ a, b \in \mathbb{C}, \ \tau \in \mathbb{R}$$

and $f \prec g$. If $L_\gamma : H(U) \to H(U)$is the Libera generalized integral operator defined by (2) then $L_\gamma^n(f)$ and $L_\gamma^n(g)$ are bounded, for all $n \in \mathbb{N}^*$, where

$$L_\gamma^n = \underbrace{L_\gamma \circ L_\gamma \circ \ldots \circ L_\gamma}_{n}.$$

Proof. Let F , I and G be operators defined by

$$F[f](z) = \frac{\gamma + 1}{z^{\gamma - 1}} f(z),$$

$$I[f](z) = \frac{1}{z} \int_0^z f(t)\, dt,$$

$$G[f](z) = f(z) \cdot z^{\gamma - 1}.$$

A simple computation shows that L_γ can be written as:

$$L_\gamma = F \circ I \circ G. \tag{3.1}$$

From Lemma 2.1, we have that if g is convex function and is not of the form

$$g(z) = a + \frac{b}{1 - ze^{i\tau}}\,,\ a, b \in \mathbb{C}\,,\ \tau \in \mathbb{R}$$

then there exists $\varepsilon = \varepsilon(g) > 0$ such that

$$g \in H^{1+\varepsilon}.$$

We have

$$M_p(r, G(f)) = \left(\frac{1}{2\pi} \int_0^{2\pi} \left| f(re^{i\theta}) \cdot (re^{i\theta})^{\gamma-1} \right|^p d\theta \right)^{\frac{1}{p}} =$$

$$= k \left(\frac{1}{2\pi} \int_0^{2\pi} \left| f(re^{i\theta}) \right|^p d\theta \right)^{\frac{1}{p}} = kM_p(r, f), \tag{3.2}$$

where k constant.

Hence, $G(f)$ and f have the same Hardy spaces, that is $G(f) \in H^{1+\varepsilon}$.

From theorem of Hardy-Littlewood we have $\int_0^z g(t)\,dt \in H^\infty$. But $\int_0^z g(t)\,dt$ and $I[g]$ have the same Hardy spaces. Hence $(I \circ G)(g) \in H^\infty$. Also, f and $F(f)$ have the same Hardy spaces. Hence we obtain $(F \circ I \circ G)(g) \in H^\infty$, and $L_\gamma(g) \in H^\infty$. Because $f \prec g$, from subordination theorem of Littlewood we have $f \in H^{1+\varepsilon}$, $\varepsilon > 0$. Hence $G(f) \in H^{1+\varepsilon}$ and $I(G(f)) \in H^\infty$. Analogous $F(I(G(f))) \in H^\infty$. Finally $L_\gamma(f) \in H^\infty$.

Because $I(L_\gamma(f)) \in H^\infty$ we have $I(G(L_\gamma)) \in H^\infty$ and $F(I(G(L_\gamma(f))) \in H^\infty$. Hence $L^2_\gamma(f) \in H^\infty$. Applying the theorem of induction we obtain the result.

Corollary 3.2. *Let Let $f, g \in H(U), g$ convex, g is not of the form*

$$g(z) = a + \frac{b}{1 - ze^{i\tau}}\,,\ a, b \in \mathbb{C}\,,\ \tau \in \mathbb{R}$$

and $f \prec g$. Then, for the Libera integral operator defined by (1), we have that $L^n(f)$ and $L^n(g)$ are bounded in U for all $n \in \mathbb{N}^*$, where

$$L^n = \underbrace{L \circ L \circ \ldots \circ L}_{n}.$$

Indeed, if we let $\gamma = 1$ in Theorem 3.1 we obtain the result.

In other words, L maps K into the class of bounded functions without exception. Also, the class of functions which are subordinate of convex functions is maps into the class of bounded functions.

Remark 3.3. For $\gamma = 0$ the Libera generalized integral operator becomes the Alexander integral operator. In a similar way we can show that the results is true for the Alexander operator.

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