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About some inequalities

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Abstract. In this article we prove some inequalities of Simpson's type and we present some applications.

1. Introduction

The following result was proved in [1]:

Theorem 1.1. *Let $f : [a, b] \to \mathbb{R}$ be a mapping with bounded variation on $[a, b]$. Then we have the inequality*

$$(1.1)\quad \Big|\int_a^b f(x)dx - \frac{b-a}{3}\Big[\frac{f(a)+f(b)}{2} + 2f\Big(\frac{a+b}{2}\Big)\Big]\Big| \leq \frac{1}{3}(b-a)\bigvee_a^b(f),$$

where $\bigvee_a^b(f)$ denote the total variation of f on the interval $[a, b]$. The constant $\frac{1}{3}$ is the best possible.

Corollary 1.1. *Suppose that $f : [a, b] \to \mathbb{R}$ is a differentiable mapping whose derivative is coutinuous on (a, b) and $\|f'\|_1 = \int_a^b |f'(x)|dx < \infty$. Then we have the inequality*

$$(1.2)\quad \Big|\int_a^b f(x)dx - \frac{b-a}{3}\Big[\frac{f(a)+f(b)}{2} + 2f\Big(\frac{a+b}{2}\Big)\Big]\Big| \leq \frac{1}{3}\|f'\|_1(b-a).$$

2. Main results

Theorem 2.1. *Let $f : [a, b] \to \mathbb{R}$ be a mapping with bounded variation on $[a, b]$. Then we have the inequality*

$$(2.1)l\Big|\int_a^b f(x)dx \quad - \quad \Big[2t\frac{f(a)+f(b)}{2} + (1-2t)f\Big(\frac{a+b}{2}\Big)\Big](b-a)\Big| \leq$$

$$(2.2)\qquad \leq \quad \frac{b-a}{2}\bigvee_a^b(f)\max\{2t, |1-2t|$$

for all $t \in [0, 1]$.

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Proof: We define the function

$$s_t\left(x\right)=\begin{cases} x-\Big[\left(1-t\right)a+tb\Big], & x\in\Big[a,\dfrac{a+b}{2}\Big)\\ x-\Big[ta+(1-t)b\Big], & x\in\Big[\dfrac{a+b}{2},b\Big]\end{cases},\ t\in[0,1].$$

Using the integration by parts formula for Riemann-Stieltjes integral, for $t\in[0,1]$ we have

$$\int_a^b s_t(x)df(x)=\int_a^{\frac{a+b}{2}} s_t(x)df(x)+\int_{\frac{a+b}{2}}^b s_t(x)df(x)=$$

$$=\int_a^{\frac{a+b}{2}}\Big\{x-\Big[(1-t)a+tb\Big]\Big\}df(x)+$$

$$+\int_{\frac{a+b}{2}}^b\Big\{x-\Big[ta+(1-t)b\Big]\Big\}df(x)=$$

$$=\Big(\Big\{x-\Big[(1-t)a+tb\Big]\Big\}f(x)\Big|_a^{\frac{a+b}{2}}-\int_a^{\frac{a+b}{2}}f(x)dx\Big)+$$

$$+\Big(\Big\{x-\Big[ta+(1-t)b\Big]\Big\}f(x)\Big|_{\frac{a+b}{2}}^b-\int_{\frac{a+b}{2}}^b f(x)dx\Big),$$

therefore

(2.3)
$$\int_a^b s_t(x)df(x)=\Big[2t\frac{f(a)+f(b)}{2}+(1-2t)f\Big(\frac{a+b}{2}\Big)\Big](b-a)-\int_a^b f(x)dx,$$

for all $t\in[0,1]$.

Now assume that $\Delta_n: a=x_0^{(n)}<x_1^{(n)}<\ldots<x_{k_n}^{(n)}=b$ is a sequence of divisions with $\lim\limits_{n\to\infty}\|\Delta_n\|=0$, where

$$\|\Delta_n\|=\max_{i\in\{1,2,\ldots,k_n\}}(x_i^{(n)}-x_{i-1}^{(n)})\text{ and }\xi_i^{(n)}\in[x_{i-1}^{(n)},x_i^{(n)}],$$
$$i\in\{1,2,\ldots,k_n\}.$$

For $t\in[0,1]$, then

$$\Big|\int_a^b s_t(x)df(x)\Big|=\Big|\lim_{n\to\infty}\sum_{i=1}^{k_n}s_t(\xi_i^{(n)})\Big[f(x_i^{(n)})-f(x_{i-1}^{(n)})\Big]\Big|\le$$

$$\le\lim_{n\to\infty}\sum_{i=1}^{k_n}\Big|s_t(\xi_i^{(n)})\Big|\Big|f(x_i^{(n)})-f(x_{i-1}^{(n)})\Big|\le\max_{x\in[a,b]}|s_t(x)|\sup_{\Delta_n}\sum_{i=1}^{k_n}\Big|f(x_i^{(n)})-f(x_{i-1}^{(n)})\Big|,$$

so

(2.4)
$$\Big|\int_a^b s_t\left(x\right)df\left(x\right)\Big|\le\max_{x\in[a,b]}\left|s_t\left(x\right)\right|\overset{b}{\underset{a}{V}}\left(f\right)$$

for all $t \in [0,1]$.
Taking into account the fact that s_t is monotonic nondecreasing on the intervals $\left[a, \frac{a+b}{2}\right)$ and $\left[\frac{a+b}{2}, b\right]$, and

$$s_t(a) = -t(b-a),$$

$$s_t\Big(\frac{a+b}{2} - 0\Big) = \frac{(1-2t)(b-a)}{2},$$

$$s_t\Big(\frac{a+b}{2} + 0\Big) = -\frac{(1-2t)(b-a)}{2},$$

$$s_t(b) = t(b-a),$$

we obtain that

$$\max_{x\in[a,b]} |s_t(x)| = \frac{b-a}{2} \max\{2t, |1-2t|\} \text{ for all } t \in [0,1]. \tag{2.5}$$

Now using the inequalities (2.2) - (2.4), we get the desired result from (2.1).

Remark. Choosing $t = \frac{1}{6}$ in Theorem 2.1 we get Theorem 1.1.

Remark. Suppose that $f : [a,b] \to \mathbb{R}$ is a differentiable mapping whose derivative is continuous on (a,b). Then

$$\int_a^b |f'(x)|dx = \bigvee_a^b (f).$$

For choosing $t = \frac{1}{6}$ in Theorem 2.1. we obtain Corollary 1.1.

Corollary 2.1. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on $[a,b]$ and whose derivative $f' : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$. Then we have the inequality*

$$\Big| \int_a^b f(x)dx - \Big[2t\frac{f(a)+f(b)}{2} + (1-2t)f\Big(\frac{a+b}{2}\Big)\Big](b-a)\Big| \leq \tag{2.6}$$

$$\leq \frac{b-a}{2} \bigvee_a^b (f) \max\{2t, |1-2t|\} \leq \frac{(b-a)^2}{2} \|f'\|_\infty \max\{2t, |1-2t|\}$$

for all $t \in [0,1]$.

Proof: Taking in acount that

$$\int_a^b |f'(x)|dx = \bigvee_a^b (f),$$

$$\int_a^b |f'(x)|dx \leq (b-a)\sup_{x\in[a,b]} |f'(x)| = = (b-a)\|f'\|_\infty$$

and then of Theorem 2.1.

Corollary 2.2. *Suppose that $f : [a, b] \to \mathbb{R}$ is a differentiable mapping whose derivative is continuous on $[a, b]$. Then we have the inequalities*

$$\Big| \int_a^b f(x)dx - (b-a) f\Big(\frac{a+b}{2}\Big) \Big| \le \frac{b-a}{2} \bigvee_a^b (f) \le \frac{(b-a)^2}{2} \|f'\|_\infty, \tag{2.7}$$

$$\Big| \int_a^b f(x)dx - (b-a) \frac{f(a)+f(b)}{2} \Big| \le \frac{b-a}{2} \bigvee_a^b (f) \le \frac{(b-a)^2}{2} \|f'\|_\infty \tag{2.8}$$

and

$$\Big| \int_a^b f(x)dx + \Big[f\Big(\frac{a+b}{2}\Big) - 2\frac{f(a)+f(b)}{2} \Big](b-a) \Big| \le \tag{2.9}$$

$$\le \quad \frac{b-a}{2} \bigvee_a^b (f) \le \frac{(b-a)^2}{2} \|f'\|_\infty .$$

Proof: If we choose in (2.5) $t = 0, t = \frac{1}{2}$, respectively $t = 1$, we get (2.6) - (2.8).

Corollary 2.3. *Suppose that $f : [a, b] \to \mathbb{R}$ is a differentiable mapping whose derivative is continuous on $[a, b]$. Then*

$$\Big| \int_a^b f(x)dx - \Big[2t\frac{f(a)+f(b)}{2} + (1-2t) f\Big(\frac{a+b}{2}\Big) \Big](b-a) \Big| \le \tag{2.10}$$

$$\le (b-a) \bigvee_a^b (f) \le (b-a)^2 \|f'\|_\infty$$

for all $t \in [0, 1]$.

Proof: Let function $g : [0, 1] \to \mathbb{R}, g(t) = 2t - |1 - 2t|, t \in [0, 1]$. We have

$$g(t) = \begin{cases} 4t - 1, & t \in \left[0, \frac{1}{2}\right) \\ 1, & t \in \left[\frac{1}{2}, 1\right] \end{cases},$$

from where

$$\max\{2t, |1-2t|\} = \begin{cases} 1 - 2t, & t \in \left[0, \frac{1}{4}\right) \\ 2t, & t \in \left[\frac{1}{4}, 1\right] \end{cases}$$

and $\max\limits_{t \in [0,1]} \max\{2t, |1 - 2t|\} = 2$. Taking in account relation (2.5), we obtain the (2.9) inequality.

Corollary 2.4. *Suppose that $f : [a,b] \to \mathbb{R}$ is a differentiable mapping whose derivative is countinuous on $[a,b]$. Then*

$$\begin{aligned}(2.11)\qquad \max\ & \Big\{\Big|\int_a^b f(x)dx-(b-a)f\Big(\frac{a+b}{2}\Big)\Big|,\Big|\int_a^b f(x)dx-\\ - & \Big[f(a)+f(b)-f\Big(\frac{a+b}{2}\Big)\Big](b-a)\Big|\Big\}\le\\ \le & (b-a)\bigvee_a^b(f)\le(b-a)^2\|f'\|_\infty.\end{aligned}$$

Proof: Function $h : [0,1] \to \mathbb{R}$,

$$h(t)=\Big[f\Big(\tfrac{a+b}{2}\Big)-\tfrac{f(a)+f(b)}{2}\Big](b-a)2t++\int_a^b f(x)dx-f\Big(\frac{a+b}{2}\Big)(b-a)$$

is at most 1 degree, so touches its extremes in $t = 0$ and $t = 1$. Then to be applyed Corollary 2.3.

The following approximation of the integral $\int_a^b f(s)dx$ holds:

Theorem 2.2. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping whose derivative is continuous on $[a,b]$. If $I_n : a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$ is a partition of $[a,b]$ and $h_i = x_{i+1} - x_i, i \in \{0,1,\ldots,n-1\}$, then*

$$(2.12)\qquad \int_a^b f(x)dx = A(I_n,t,f)+R(I_n,t,f)$$

where

$$(2.13)\quad A(I_n,t,f)=2t\sum_{i=0}^{n-1}\frac{f(x_i)+f(x_{i+1})}{2}h_i+(1-2t)\sum_{i=0}^{n-1}f\Big(\frac{x_i+x_{i+1}}{2}\Big)h_i$$

for all $t \in [0,1]$ and the remainder term satisfies the estimation

$$(2.14)\qquad |R(I_n,t,f)|\le\frac{1}{2}\max\{2t,|1-2t|\}\sum_{i=0}^{n-1}\bigvee_{x_i}^{x_{i+1}}(f)\cdot h_i\le$$

$$\le\frac{1}{2}\max\{2t,|1-2t|\}\sum_{i=0}^{n-1}\|f'\|_\infty^{(i)}h_i^2\le\frac{1}{2}\|f'\|_\infty\max\{2t,|1-2t|\}\sum_{i=0}^{n-1}h_i^2$$

for all $t \in [0,1]$, where $\|f'\|_\infty^{(i)} = \sup_{x\in[x_i,x_{i+1}]} |f'(x)|$, $i \in \{0,1,2,...,n-1\}$.

Proof: Applying Corollary 2.1 on the interval $[x_i, x_{i+1}], i \in \{0,1,\ldots,n-1\}$ we get

$$-\frac{1}{2}\max\{2t,|1-2t|\}\|f'\|_\infty^{(i)}h_i^2 \le -\frac{1}{2}\max\{2t,|1-2t|\}\bigvee_{x_i}^{x_{i+1}}(f)h_i \le$$
$$\int_{x_i}^{x_{i+1}} f(x)dx- -\Big[2t\frac{f(x_i)+f(x_{i+1})}{2}+(1-2t)f\Big(\frac{x_i+x_{i+1}}{2}\Big)\Big]h_i \le$$
$$\frac{1}{2}\max\{2t,|1-2t|\}\bigvee_{x_i}^{x_{i+1}}(f)h_i \le \le \frac{1}{2}\max\{2t,|1-2t|\}\|f'\|_\infty^{(i)}h_i^2, \text{ for all } t\in[0,1].$$

Summing over i from o to $n-1$, we get estimation (2.13).

Corollary 2.5. *In condition of the Theorem 2.2., we have the inequalities*

$$|R(I_n,t,f)| \le \sum_{i=0}^{n-1}\bigvee_{x_i}^{x_{i+1}}(f)\ h_i \le \sum_{i=0}^{n-1}\|f'\|_\infty^{(i)}\ h_i^2 \le \|f'\|_\infty\sum_{i=0}^{n-1}h_i^2 \tag{2.15}$$

for all $t\in[0,1]$ *and*

$$\max\Big\{|R(I_n,0,f)|,|R(I_n,1,f)|\Big\} \le \sum_{i=0}^{n-1}\bigvee_{x_i}^{x_{i+1}}(f)\ h_i \le \tag{2.16}$$
$$\le \sum_{i=0}^{n-1}\|f'\|_\infty^{(i)}h_i^2 \le \|f'\|_\infty\sum_{i=0}^{n-1}h_i^2.$$

Proof: Taking in account that $\max\limits_{t\in[0,1]}\max\{2t,|1-2t|\}=2$, then of Theorem 2.2. and of Corollary 2.4.

References

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