CARPATHIAN J. MATH.
19 (2003), No. 2, 141 - 146

On the mean square error of the bivariate operator of Stancu type

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ABSTRACT. In this paper we defined the Stancu bivariate operator obtained by the tensorial product of parameytric extensions (1.5) and (1.6). In the last part of the paper is established quantitative estimations of a function $f \in C([0,1]\times[0,1])$ by the polynomial (1.7)

1. INTRODUCTION

In 1969 D.D. Stancu has introduced in [4] the following generalization of the Bernstein operator

$$(1.1)\qquad (P_m^{(\alpha,\beta)} f\,)(x) = \sum_{k=0}^{m} p_{m,k}(x)\, f\,\left(\frac{k+\alpha}{m+\beta}\right)$$

where $p_{m,k}(x)$ are the basic polinomials

$$(1.2)\qquad p_{m,k} = \left(\begin{array}{c} m \\ k \end{array}\right) x^k(1-x)^{m-k},$$

and α, β are real parameters, independently of m, satisfying the relations $0 \le \alpha \le \beta$.

For $\alpha = \beta = 0$ we obtain the classical Bernstein polynomial $B_m f$, which coincides with function f at $x_0 = 0$ and $x_m = 1$.

If $0 < \alpha \neq \beta$ the polynomial does not coincide at any node with the function f. If $\alpha = 0$ and $\alpha \neq \beta$ then it coincides with f at $x_0 = 0$, while if $0 < \alpha = \beta$ then it coincides with f at $x_m = 1$.

Let $0 \le \alpha_1 \le \beta_1$, $0 \le \alpha_2 \le \beta_2$ and the Stancu operators $P_m^{(\alpha_1,\beta_1)}, P_n^{(\alpha_2,\beta_2)} : C([0,1]) \to C([0,1])$ given by

Received: 09.08.2003; In revised form: 03.03.2004
2000 *Mathematics Subject Classification.* 41A10, 41A25, 41A36.
Key words and phrases. *Bivariate Stancu operator, mean square error.*

$$(1.3)\qquad (P_m^{(\alpha_1,\beta_1)}f\ )(x)=\sum_{k=0}^{m}p_{m,k}(x)\,f\ \left(\frac{k+\alpha_1}{m+\beta_1}\right),$$

for all $x\in[0,1]$, and $f\in C([0,1])$

$$(1.4)\qquad (P_n^{(\alpha_2,\beta_2)}g\ )(y)=\sum_{j=0}^{n}p_{n,j}(y)\,f\ \left(\frac{j+\alpha_2}{n+\beta_2}\right),$$

for all $y\in[0,1]$ and $g\in C([0,1])$, where $p_{m,k}$ and $p_{n,j}$ are the basic Bernstein polynomials of degree m, respectively n.

The parametric extensions are

$$_xP_m^{(\alpha_1,\beta_1)},_y P_n^{(\alpha_2,\beta_2)}:C([0,1]\times[0,1])\to C([0,1]\times[0,1])$$

$$(_xP_m^{(\alpha_1,\beta_1)}f\ )(x,y)=\sum_{k=0}^{m}p_{m,k}(x)\,f\ \left(\frac{k+\alpha_1}{m+\beta_1},y\right),$$

where $(x,y)\in[0,1]\times[0,1]$ and $\forall f\in C([0,1]\times[0,1])$

$$(_yP_n^{(\alpha_2,\beta_2)}f\ )(x,y)=\sum_{j=0}^{n}p_{n,j}(y)\,g\ \left(x,\frac{k+\alpha_2}{m+\beta_2}\right),$$

where $(x,y)\in[0,1]\times[0,1]$ and $\forall g\in C([0,1]\times[0,1])$

These operators are linear and positive, which commute.

The Stancu bivariate operator is defined by the tensorial product of parametric extensions and we obtain

$$P_{m,n}^{(\alpha_1,\beta_1;\alpha_2\beta_2)}:C([0,1]\times[0,1])\to C([0,1]\times[0,1])$$

$$(1.5)\quad (P_{m,n}^{(\alpha_1,\beta_1;\alpha_2\beta_2)}f)(x,y)=\sum_{k=0}^{m}\sum_{j=0}^{n}p_{m,k}(x)p_{n,j}(y)f\left(\frac{k+\alpha_1}{m+\beta_1},\frac{j+\alpha_2}{n+\beta_2}\right)$$

This operator has also been studied by F. Stancu in [7]. It is part of a broader class of bidimensions positiv liniar operators studied by D. Bărbosu in [2] and [3].

For the test functions e_{ij}, $i=\overline{0,2}$, $j=\overline{0,2}$ where $e_{ij}(t,s)=t^i s^j$, we have

$$\left(P_{m,n}^{(\alpha_1,\beta_1;\alpha_2,\beta_2)}\,e_{00}\right)=e_{00};$$

$$\left(P_{m,n}^{(\alpha_1,\beta_1;\alpha_2,\beta_2)}\,e_{10}\right)(x,y)=x+\frac{\alpha_1-\beta_1 x}{m+\beta_1};$$

$$\left(P_{m,n}^{(\alpha_1,\beta_1;\alpha_2,\beta_2)}\,e_{01}\right)(x,y)=y+\frac{\alpha_2-\beta_2 y}{n+\beta_2};$$

$$\left(P_{m,n}^{(\alpha_1,\beta_1;\alpha_2,\beta_2)}\,e_{20}\right)(x,y)=x^2+\frac{mx(1-x)}{(m+\beta_1)^2}+\frac{(\alpha_1-\beta_1 x)(2mx+\beta_1 x+\alpha_1)}{(m+\beta_1)^2};$$

$$\left(P_{m,n}^{(\alpha_1,\beta_1;\alpha_2,\beta_2)}\,e_{02}\right)(x,y)=y^2+\frac{ny(1-y)}{(n+\beta_2)^2}+\frac{(\alpha_2-\beta_2 y)(2ny+\beta_2 x+\alpha_2)}{(n+\beta_2)^2}.$$

2. The mean square errors of the operator $P_{m,n}^{(\alpha_1,\beta_1;\alpha_2,\beta_2)}$

For the Stancu bivariate operator $P_{m,n}^{(\alpha_1,\beta_1;\alpha_2,\beta_2)}$ the mean square errors are given by the relations

$$\begin{aligned} e_{m,n}^{(2,0)}(x,y;\alpha_1,\beta_1) &= P_{m,n}^{(\alpha_1,\beta_1;\alpha_2,\beta_2)}((t-x)^2;x,y);\\ e_{m,n}^{(0,2)}(x,y;\alpha_2,\beta_2) &= P_{m,n}^{(\alpha_1,\beta_1;\alpha_2,\beta_2)}((s-y)^2;x,y). \end{aligned} \tag{2.1}$$

According to (18) we have

$$\begin{aligned} e_{m,n}^{(2,0)}(x,y;\alpha_1,\beta_1) &= \frac{mx(1-x)+(\alpha_1 x-\beta_1)^2}{(m+\beta_1)^2};\\ e_{m,n}^{(0,2)}(x,y;\alpha_2,\beta_2) &= \frac{my(1-y)+(\alpha_2 y-\beta_2)^2}{(n+\beta_2)^2}. \end{aligned} \tag{2.2}$$

In order to see how well a function $f\in C([0,1]\times[0,1])$ can be approximated by the polynomial (1.7), we must find the maximal value of (2.10) on the interval $[0,1]\times[0,1]$.

Theorem 2.1. *If* $m > \beta_1^2$ *and* $n > \beta_2^2$ *, then the maximal values on* $[0,1]\times[0,1]$ *of the mean square errors (2.9) can be written as follow*

$$M^{2,0}(m;\alpha_1,\beta_1) = \frac{m}{4(m+\beta_1)^2}\left[1+\frac{(\beta_1-2\alpha_1)^2}{m-\beta_1^2}\right]; \tag{2.3}$$

$$M^{0,2}(n;\alpha_2,\beta_2) = \frac{n}{4(n+\beta_2)^2}\left[1+\frac{(\beta_2-2\alpha_2)^2}{n-\beta_2^2}\right].$$

Corollary 2.1. *The least maximal values of relations (2.11) are attained for* $\beta_1 = 2\alpha_1,\ \beta_2 = 2\alpha_2$ *and they are:*

$$\nu^{2,0}(\alpha_1) = \frac{m}{4(m+2\alpha_1)^2} \le \frac{1}{4m} = \nu^{2,0}(0); \tag{2.4}$$

$$\nu^{0,2}(\alpha_2) = \frac{n}{4(n+2\alpha_2)^2} \le \frac{1}{4n} = \nu^{0,2}(0).$$

In this case the approximating polynomial (1.7) is

$$\left(P_{m,n}^{(\alpha_1,2\alpha_1;\alpha_2,2\alpha_2)} f\right)(x,y) = \left(S_{m,n}^{(\alpha_1,\alpha_2)}\right)(x,y) =$$

$$\sum_{k=0}^{m}\sum_{j=0}^{n} p_{m,k}(x)p_{n,j}(y)\, f\left(\frac{k+\alpha_1}{m+2\alpha_1},\frac{j+\alpha_2}{n+2\alpha_2}\right), \quad (\alpha_1 \ge 0,\quad \alpha_2 \ge 0). \tag{2.5}$$

3. Quantitative estimation of approximation

We establish some estimates of the order of approximation of a function $f \in C([0,1]\times[0,1])$ by the polynomial (1.7). Because the constants are reproduced and using the result of [3] we obtain:

$$\begin{aligned}\left|f(x,y)-\left(P_{m,n}^{(\alpha_1,\beta_1;\alpha_2,\beta_2)} f\right)(x,y)\right| \le & \left[1+\frac{1}{\delta_1}\sqrt{P_{m,n}^{(\alpha_1,\beta_1;\alpha_2,\beta_2)}((t-x)^2;x,y)}+\right.\\ & + \frac{1}{\delta_2}\sqrt{P_{m,n}^{(\alpha_1,\beta_1;\alpha_2,\beta_2)}((s-y)^2;x,y)}\\ & + \frac{1}{\delta_1\delta_2}\sqrt{P_{m,n}^{(\alpha_1,\beta_1;\alpha_2,\beta_2)}((t-x)^2;x,y)}\\ & \left.\cdot\sqrt{P_{m,n}^{(\alpha_1,\beta_1;\alpha_2,\beta_2)}((s-y)^2;x,y}\right]\omega_1(f;\delta_1,\delta_2),\end{aligned} \tag{3.1}$$

where ω_1 is the first order modulus of continuity and $\delta_1 > 0,\ \delta_2 > 0$.

Taken into account the relations (2.1) and (2.2) we have:

$$\left|f(x,y)-\left(P_{m,n}^{(\alpha_1,\beta_1;\alpha_2,\beta_2)}f\right)(x,y)\right|\le\left[1+\frac{1}{\delta_1}\sqrt{e_{m,n}^{2,0}(x,y;\alpha_1,\beta_1)}+\right.$$

$$+\frac{1}{\delta_2}\sqrt{e_{m,n}^{0,2}(x,y;\alpha_2,\beta_2)}+\frac{1}{\delta_1\delta_2}\sqrt{e_{m,n}^{2,0}(x,y;\alpha_1,\beta_1)}\cdot \tag{3.2}$$

$$\left.\cdot\sqrt{e_{m,n}^{0,2}(x,y;\alpha_2,\beta_2)}\right]\ \omega_1\left(f;\delta_1,\delta_2\right).$$

Theorem 3.1. *If $m>\beta_1^2$, and $n>\beta_2^2$ then we have the inequality*

$$\|f\ -\ P_{m,n}^{(\alpha_1,\beta_1;\alpha_2,\beta_2)}f\Big\|\le\left[1+\frac{1}{2}\sqrt{1+\frac{(\beta_1-2\alpha_1)^2}{m-\beta_1^2}}\right.$$

$$+\ \frac{1}{2}\sqrt{1+\frac{(\beta_2-2\alpha_2)^2}{n-\beta_2^2}}+\frac{1}{4}\sqrt{1+\frac{(\beta_1-2\alpha_1)^2}{m-\beta_1^2}}$$

$$\cdot\ \left.\sqrt{1+\frac{(\beta_2-2\alpha_2)^2}{n-\beta_2^2}}\right]\omega_1\left(f;\sqrt{\frac{m}{(m+\beta_1)^2}},\sqrt{\frac{n}{(n+\beta_2)^2}}\right). \tag{3.3}$$

For $\alpha_1=\beta_1=0$ and $\alpha_2=\beta_2=0$ the inequality (3.3) is the inequality of the Tiberiu Popoviciu:

$$\|\ f-B_{m,n}f\ \|\le\frac{9}{4}\omega_1\ \left(\ f;\sqrt{\frac{1}{m}},\sqrt{\frac{1}{n}}\right). \tag{3.4}$$

For $\beta_1=2\alpha_1$ and $\beta_2=2\alpha_2$ the inequality (3.3) becams:

(3.5)
$$\left\|\ f-P_{m,n}^{(\alpha_1,2\alpha_1;\alpha_2,2\alpha_2)}f\ \right\|\le\frac{9}{4}\omega_1\left(\ f;\sqrt{\frac{m}{(m+2\alpha_1)^2}},\sqrt{\frac{n}{(n+2\alpha_2)^2}}\right).$$

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