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Semilinear operator equations in real Hilbert spaces with Lipschitz nonlinearity

DINU TEODORESCU

ABSTRACT. In this paper we establish an existence and uniqueness result for the semilinear equation $Au + F(u) = f$, making only the supposition that the nonlinearity F is a Lipschitz operator. We use in this study a contractive method based on the Picard-Banach fixed point theorem(the method is frequently used in the study of the variational inequalities-see the proof of the Lions-Stampacchia theorem in [4], minimization methods for convex functionals in [3], a.s.o.).

1. INTRODUCTION

In [1] and [2] H. Amann studies semilinear equations of the form $Au - F(u) = 0$ in a real Hilbert space H, where the nonlinearity $F : H \longrightarrow H$ is a Gateaux differentiable gradient operator which interacts suitably with the spectrum of the linear operator A. In [6] C. Mortici obtains an existence and uniqueness result for the semilinear equation $Au + F(u) = 0$, where the nonlinearity F is a strongly monotone Lipschitz operator.

Let H be a real Hilbert space endowed with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\|\cdot\|$.The absolute value of the real number α will be denote by $|\alpha|$.

In [1] and [2] it is studied semilinear equations of the form $Au-F(u) = 0$, where $A : D(A) \subseteq H \longrightarrow H$ is a self-adjoint linear operator with the resolvent set $R(A)$ and $F : H \longrightarrow H$ is a Gateaux differentiable gradient operator. If there exist real numbers $a < b$ such that $[a, b] \subset R(A)$ and

$$a \leq \frac{\langle F(u) - F(v), u - v \rangle}{\|u - v\|^2} \leq b$$

for all $u, v \in H$ with $u \neq v$(i.e. F interacts with the spectrum of A), then it proves in [2] that the equation $Au - F(u) = 0$ has exactly one solution.

In [6] is considered the equation $Au+F(u) = 0$, where $A : D(A) \subseteq H \longrightarrow H$ is a linear maximal monotone operator and the nonlinearity $F : H \longrightarrow H$ is a strongly monotone Lipschitz operator. It shows that, under these assumptions, the equation $Au + F(u) = 0$ has an unique solution.

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Let $F : H \longrightarrow H$ be a Lipschitz nonlinear operator with constant $M > 0$, i.e.

$$\|F(x) - F(y)\| \leq M \|x - y\|$$

for all $x, y \in H$ and $A : D(A) \subseteq H \longrightarrow H$ be a linear operator such that $D(A)$ is a linear subspace of H.

For $f \in H$ we consider the semilinear equation

$$Au + F(u) = f. \tag{1.1}$$

We will prove in this note that the supposition "F is a Lipschitz operator" is sufficiently for obtaining existence and uniqueness results for the equation (1.1).

In section 3 some applications are given.

2. The results

We suppose that A is a strongly positive operator i.e. there exists $c > 0$ such that

$$\langle Ax, x\rangle \geq c \|x\|^2$$

for all $x \in D(A)$ and $c > M$, $Rg(F) \subseteq Rg(A)$

$$Rg(F) = \{F(x) \mid x \in H\} \quad , \quad Rg(A) = \{Ax \mid x \in D(A)\},$$

$f \in Rg(A)$, ($Rg(A)$ is a linear subspace of H because A is linear).

Using the Cauchy-Schwarz inequality, we obtain

$$c\|x\|^2 \leq \langle Ax, x\rangle = |\langle Ax, x\rangle| \leq \|Ax\| \cdot \|x\|,$$

for all $x \in D(A)$. It results that

$$\|Ax\| \geq c\|x\|,$$

for all $x \in D(A)$. Consequently there exists $A^{-1} : Rg(A) = H_1 \longrightarrow H$ which is linear and continuous, $A^{-1} \in L(H_1, H)$, the normed space of all linear and continuous operators from $H_1$to H. Moreover

$$\left\|A^{-1}\right\|_{L(H_1,H)} \leq \frac{1}{c},$$

where

$$\left\|A^{-1}\right\|_{L(H_1,H)} = \sup\{\left\|A^{-1}v\right\| \mid v \in H_1, \|v\| \leq 1\}.$$

Now the equation (1) can be equivalently written as

$$(I + A^{-1}F)u = A^{-1}f, \tag{2.1}$$

where I is the identity of H.With the notations $V = I + A^{-1}F$ and $g = A^{-1}f$ the equation (2.1) becomes

$$Vu = g. \tag{2.2}$$

Proposition 1. *$V : H \longrightarrow H$ is a strongly monotone Lipschitz operator, i.e.there exist $\alpha, \beta > 0$ such that*

$$\|Vx - Vy\| \leq \alpha \|x - y\| \tag{2.3}$$

$$\langle Vx - Vy, x - y\rangle \geq \beta \|x - y\|^2 \textit{ for all } x, y \in H. \tag{2.4}$$

Proof. Using the properties of A and F, the Cauchy-Schwarz inequality and the fact that H is a real Hilbert space, we obtain

$$\|Vx - Vy\| = \left\|x + A^{-1}Fx - y - A^{-1}Fy\right\| \leq \|x - y\| + \left\|A^{-1}(F(x) - F(y))\right\|$$
$$\leq \|x - y\| + \left\|A^{-1}\right\|_{L(H_1,H)} \cdot \|F(x) - F(y)\| \leq \left(1 + \frac{M}{c}\right)\|x - y\|$$

for all $x, y \in H$. Therefore (2.3) is true with $\alpha = 1 + \dfrac{M}{c}$. Also we have

$$-\left\langle A^{-1}Fx - A^{-1}Fy, x - y\right\rangle = -\left\langle A^{-1}(F(x) - F(y)), x - y\right\rangle \leq$$
$$\leq \left|\left\langle A^{-1}(F(x) - F(y)), x - y\right\rangle\right| \leq$$
$$\leq \left\|A^{-1}(F(x) - F(y))\right\| \cdot \|x - y\| \leq \frac{M}{c}\|x - y\|^2$$

and then

$$\langle Vx - Vy, x - y\rangle = \left\langle x + A^{-1}Fx - y - A^{-1}Fy, x - y\right\rangle =$$
$$= \|x - y\|^2 + \left\langle A^{-1}Fx - A^{-1}Fy, x - y\right\rangle$$
$$\geq \|x - y\|^2 - \frac{M}{c}\|x - y\|^2 = \left(1 - \frac{M}{c}\right)\|x - y\|^2,$$

for all $x, y \in H$. It follows that (2.4) is true with $\beta = 1 - \dfrac{M}{c} > 0$. □

For $\gamma > 0$ we consider the operator $S_\gamma : H \longrightarrow H$ defined by

$$S_\gamma u = u - \gamma(Vu - g) = (I - \gamma V)u + \gamma g.$$

Next we will use the following

Proposition 2. *There exist $\gamma > 0$ and $\lambda \in (0, 1)$ such that*

$$\|S_\gamma x - S_\gamma y\| \leq \lambda \|x - y\|, \textit{ for all } x, y \in H.$$

For proof and more details see [6], [8]. Indeed, with (2.3)-(2.4),

$$\|S_\gamma x - S_\gamma y\|^2 = \langle x - \gamma Vx - y + \gamma Vy, x - \gamma Vx - y + \gamma Vy\rangle =$$
$$= \langle x - y - \gamma(Vx - Vy), x - y - \gamma(Vx - Vy)\rangle =$$
$$= \|x - y\|^2 - 2\gamma\langle Vx - Vy, x - y\rangle + \gamma^2\|Vx - Vy\|^2$$
$$\leq (1 - 2\gamma\beta + \alpha^2\gamma^2)\|x - y\|^2,$$

for all $x, y \in H$. Now, $1 - 2\gamma\beta + \alpha^2\gamma^2 < 1$, if $\gamma \in \left(0, \dfrac{2\beta}{\alpha^2}\right)$. Consequently, for arbitrary $\gamma \in \left(0, \dfrac{2\beta}{\alpha^2}\right)$, we have

$$\|S_\gamma x - S_\gamma y\| \leq \lambda \|x - y\|,$$

for all $x, y \in H$, with $\lambda \in (0,1)$ given by $\lambda = \sqrt{1 - 2\gamma\beta + \alpha^2\gamma^2}$. $\square$

We proved that there exists $\gamma > 0$ such that S_γ is contraction from H to H. According to the Picard-Banach fixed point theorem, S_γ has an unique fixed point. Consequently, from the definition of S_γ, it results that the equation (2.2) and so the equation (1.1) has an unique solution. Now we are in position to give the following

Theorem 1. *Let $A : D(A) \subseteq H \longrightarrow H$ be linear, $F : H \longrightarrow H$ nonlinear with $Rg(F) \subseteq Rg(A)$ and assume that for some positive reals $c > M$, we have:*

(i) *F is a Lipschitz operator with constant $M > 0$,*

$$\|F(x) - F(y)\| \leq M \|x - y\|, \text{ for all } x, y \in H;$$

(ii) *A is a strongly positive operator with constant $c > 0$,*

$$\langle Ax, x\rangle \geq c \|x\|^2, \text{ for all } x \in D(A);$$

Then for each $f \in Rg(A)$, the equation $Au + F(u) = f$ is uniquely solvable.

An estimation of the solution is given by the following

Proposition 3. *For the unique solution u^* of the equation $Au + F(u) = f$, holds*

$$\|u^\star\| \leq \frac{1}{c - M}(\|f\| + \|F(0)\|). \tag{2.5}$$

Proof. We have seen that u^* also satisfies $u^* + A^{-1}Fu^* = A^{-1}f$, so

$$\|u^\star\| = \left\|A^{-1}f - A^{-1}Fu^\star\right\| = \left\|A^{-1}(f - F(u^\star))\right\| \leq$$

$$\leq \frac{1}{c}\|f - F(u^\star)\| \leq \frac{1}{c}(\|f\| + \|F(u^\star)\|).$$

On the other side,

$$\|F(u^\star)\| \leq \|F(u^\star) - F(0)\| + \|F(0)\| \leq M\|u^\star\| + \|F(0)\|.$$

Hence

$$\|u^\star\| \leq \frac{1}{c}\|f\| + \frac{M}{c}\|u^\star\| + \frac{1}{c}\|F(0)\|,$$

and consequently

$$(c - M)\|u^\star\| \leq \|f\| + \|F(0)\|. \quad \square$$

Now, let us consider the dependence of the solution of (1) on the data f.

Proposition 4. *Let $i \in \{1,2\}$ and u_i be the unique solution of the equation*

$$Au + F(u) = f_i \ \ , \quad i = 1, 2,$$

where $f_1, f_2 \in Rg(A)$. Then

$$\|u_1 - u_2\| \leq \frac{1}{c - M} \|f_1 - f_2\| . \tag{2.6}$$

Proof. We have

$$\left| \left\| A^{-1}(f_1 - f_2) \right\| - \|u_1 - u_2\| \right| \leq \left\| A^{-1}f_1 - A^{-1}f_2 - u_1 + u_2 \right\| =$$

$$\left\| (A^{-1}f_1 - u_1) - (A^{-1}f_2 - u_2) \right\| = \left\| A^{-1}Fu_1 - A^{-1}Fu_2 \right\| \leq \frac{M}{c} \|u_1 - u_2\| .$$

It results

$$\left\| A^{-1}(f_1 - f_2) \right\| - \|u_1 - u_2\| \geq -\frac{M}{c} \|u_1 - u_2\| ,$$

then

$$\left(1 - \frac{M}{c}\right) \|u_1 - u_2\| \leq \left\| A^{-1}(f_1 - f_2) \right\| \leq \frac{1}{c} \|f_1 - f_2\| . \quad \square$$

Further, we will give the sufficient conditions under the operator A for which the equation (1.1) is uniquely solvable, for all $f \in H$.

Theorem 2. *Let $A : D(A) \subseteq H \longrightarrow H$ be linear, maximal monotone, $F : H \longrightarrow H$ nonlinear satisfying the following assumptions, for some positive reals $c > M$:*

(i) F is a Lipschitz operator with constant $M > 0$,

$$\|F(x) - F(y)\| \leq M \|x - y\| ,$$

for all $x, y \in H$;

(ii) A is a strongly positive operator with constant $c > 0$,

$$\langle Ax, x \rangle \geq c \|x\|^2 ,$$

for all $x \in D(A)$.

Then for all $f \in H$, the equation $Au + F(u) = f$ has an unique solution.

Proof. Let $\omega > 0$. The equation $Au + F(u) = f$ can be equivalently written as

$$A_\omega u + F_\omega(u) = \omega f \tag{2.7}$$

where $A_\omega = I + \omega A$ and $F_\omega = -I + \omega F$. We have $Rg(A_\omega) = H$ because A is maximal monotone. Also we obtain

$$\|F_\omega(x) - F_\omega(y)\| \leq (1 + \omega M) \|x - y\| ,$$

for all $x, y \in H$ and

$$\langle A_\omega x, x \rangle = \|x\|^2 + \omega \langle Ax, x \rangle \geq (1 + \omega c) \|x\|^2 ,$$

for all $x \in D(A) = D(A_\omega)$. Now, $1 + \omega c > 1 + \omega M$ because $c > M$ and from theorem 1, the equation (2.7) has an unique solution . $\square$

Theorem 2. *Let $A : H \longrightarrow H$ be linear, symmetric, $F : H \longrightarrow H$ nonlinear, satisfying the following conditions, for some reals $c > M$:*

(i) *F is a Lipschitz operator with constant $M > 0$,*

$$\|F(x) - F(y)\| \leq M \|x - y\|, \text{ for all } x, y \in H;$$

(ii) *A is a strongly positive operator with constant $c > 0$,*

$$\langle Ax, x\rangle \geq c \|x\|^2 \text{ for all } x \in H.$$

Then for all $f \in H$, the equation $Au + F(u) = f$ has an unique solution.

Proof. By the Hellinger-Toeplitz theorem we obtain that A is bounded, and consequently A is selfadjoint. Let us choose s in the spectrum of A. We have

$$s \geq \inf\{\langle Ax, x\rangle \mid x \in H, \|x\| = 1\}$$

We obtain now that every $\omega < 0$ is in the resolvent set of the operator A, and consequently we have

$$Rg(A - \omega I) = H,$$

for all $\omega < 0$.

Let $\delta < 0$. We write the equation $Au + F(u) = f$ in the form

$$A_\delta u + F_\delta(u) = f, \tag{2.8}$$

where $A_\delta = A - \delta I$ and $F_\delta = F + \delta I$. We have $Rg(A_\delta) = H$ and

$$\|F_\delta(x) - F_\delta(y)\| \leq (M + |\delta|) \|x - y\|,$$

for all $x, y \in H$. Also

$$\langle A_\delta x, x\rangle = \langle Ax, x\rangle - \delta \|x\|^2 \geq (c - \delta) \|x\|^2 = (c + |\delta|) \|x\|^2,$$

for all $x \in H$. Now, $c + |\delta| > M + |\delta|$ and from Theorem 1, the equation (2.8) has an unique solution. $\square$

3. Applications

(C1) Let $A : D(A) \subseteq H \longrightarrow H$ be a linear strongly positive operator, with the constant of strongly positivity $c > 1$. We assume moreover that A has a closed range.

Let $M \subseteq Rg(A)$ be a non empty set and let

$$P_{cl(conv(M))} : H \longrightarrow H$$

the projection operator on $cl(conv(M))$, the closure of the convex covering of M. It's well known that $P_{cl(conv(M))}$ is Lipschitz operator with constant equal to 1. Also we have:

$$Rg(P_{cl(conv(M))}) = cl(conv(M)) \subseteq Rg(A),$$

because $Rg(A)$ is a closed linear subspace of H. From theorem 1 we obtain:

Proposition 5. *The equation*

$$Au + P_{cl(conv(M))}u = f$$

has an unique solution for all $f \in Rg(A)$.

(C2) Let $F : H \longrightarrow H$ be a Lipschitz operator (nonlinear) with constant $M > 0$ and $c \in \mathbf{R}, c > M$. From theorem 1, for $A = cI$, we obtain:

Proposition 6. *The equation*

$$F(u) + cu = f \tag{3.1}$$

has an unique solution for all $f \in H$.

Proposition 7. *For the unique solution* $u(c, f)$ *of the equation* (3.1)*, holds*

$$\left\| u(c, f) - \frac{1}{c} f \right\| \leq \frac{1}{c - M} \|F(0)\| + \frac{M}{c^2 - cM} \|f\| . \tag{3.2}$$

Proof. From (2.5) it results that

$$\|u(c, f)\| \leq \frac{1}{c - M}(\|f\| + \|F(0)\|).$$

We have

$$c \left\| u(c, f) - \frac{1}{c} f \right\| = \|cu(c, f) - f\| = \|F(u(c, f))\| \leq$$

$$\leq \|F(u(c, f)) - F(0)\| + \|F(0)\| \leq M \|u(c, f)\| + \|F(0)\| \leq$$

$$\leq \frac{M}{c - M}(\|f\| + \|F(0)\|) + \|F(0)\| = \frac{M}{c - M} \|f\| + \frac{c}{c - M} \|F(0)\| \quad \square$$

From (3.2) we obtain that

$$\left\| u(c, f) - \frac{1}{c} f \right\| \longrightarrow 0,$$

when $c \longrightarrow \infty$. Therefore, for *large* values of c, $\frac{1}{c} f$ approximates the unique solution of the equation (3.1).

(C3) Let $f \in L^2(0, T)$. We consider the Dirichlet problem

$$\begin{array}{l} -u''(t) + au(t) - b \cdot \sin u(t) = f(t);\ t \in (0, T) \\ u(0) = u(T) = 0 \end{array}, \tag{3.3}$$

where $b \in \mathbf{R}, b > 0$ and $a \in \mathbf{R}, a > b$. We will study the problem (3.3) in the next functional background:

- $H = L^2(0, T)$;
-$A : D(A) \subseteq H \longrightarrow H$,

$$Au = -u'' + au,$$

with the domain

$$D(A) = H^2(0, T) \cap H_0^1(0, T);$$

where $F : H \longrightarrow H$,

$$F(u) = -b \cdot \sin u.$$

Let $Bu = -u''$ from H to H, defined on

$$D(B) = H^2(0,T) \cap H_0^1(0,T)$$

It is well known the fact that B is a maximal monotone operator. It results that the operator $A = B + aI$ is strongly positive with constant a and surjective. Also F is a Lipschitz operator with constant b.

From theorem 1 we obtain that the problem (3.3) has an unique solution in $H^2(0,T) \cap H_0^1(0,T)$.

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Valahia University of Targovişte, Romania
Department of Mathematics
Bd. Unirii 18, 0200 Targoviste, Romania