CARPATHIAN J. MATH.
19 (2003), No. 2, 155-160

On the stability of Picard and Mann iteration processes

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Abstract. In this paper, we establish some stability results for the Picard and Mann iteration processes considered in metric and normed linear spaces respectively. We employ the same method as in Berinde [1], but using a more general contractive definition than those in Berinde [1], Rhoades [7], Harder and Hicks [4], and Osilike [8].

1. Introduction

Let (E, d) be a complete metric space and let $T : E \to E$ be a selfmap of E. Let $F(T) = \{p^* \in E \mid Tp^* = p^*\}$ denote the set of fixed points of T. Let $\{x_n\}_{n=0}^{\infty}$ be the sequence generated by an iteration procedure involving the operator T, that is

$$x_{n+1} = f(T, x_n), n = 0, 1, 2, ... \tag{1.1}$$

where $x_0 \in E$ is the initial approximation and f is some function. Suppose $\{x_n\}_{n=0}^{\infty}$ converges to a fixed point p^* of T. Let $\{y_n\}_{n=0}^{\infty} \subset E$ and set $\epsilon_n = d(y_{n+1}, f(T, y_n)), n = 0, 1, 2, ...$ Then, the iteration procedure (1.1) is said to be T-stable or stable with respect to T if and only if $\lim_{n\to\infty} \epsilon_n = 0$ implies $\lim_{n\to\infty} y_n = p^*$.

Using this concept, Harder and Hicks [4] proved several stability results under various contractive conditions. Rhoades [6, 7] extended the results of Harder and Hicks [4] to other classes of contractive mappings. Moreover, Osilike [8] extended the results of Rhoades [7] to the following contractive definition: there exist $L \geq 0$, $a \in [0, 1)$ such that for each $x, y, \in E$,

$$d(Tx, Ty) \leq Ld(x, Tx) + ad(x, y). \tag{1.2}$$

Using (1.2) above, Osilike [8] established several stability results most of which are generalizations of the results of Rhoades [7] and Harder and Hicks [4]. In this paper, we shall establish some stability results for Picard and

Received: 02.02.2004; In revised form: 24.02.2004
2000 *Mathematics Subject Classification.* 47H10; 54H25.
Key words and phrases. *Complete metric space, fixed point iteration procedure, stability.*

Mann iteration processes using a more general contractive definition than those in Osilike [8], Rhoades [6, 7], Harder and Hicks [4] and Berinde [1]. We employ the shorter method of Berinde [1] in our proofs. This method was also used by Osilike [9]. For more details and references regarding the fixed point iteration processes and their stability, we refer to the recent monograph Berinde [10].

2. Preliminaries

Let $\{x_n\}_{n=0}^{\infty}$ be the sequence generated by the iteration procedure(1.1). Then, the Picard iteration is obtained from (1.1) if $f(T, x_n) = Tx_n$, $n = 0, 1, ...$ while the Mann iteration is obtained for

$$f(T, x_n) = (1 - \alpha_n)x_n + \alpha_n Tx_n, n = 0, 1, ..., \text{ with } \{\alpha_n\}_{n=0}^{\infty}$$

a sequence of real numbers in $[0, 1]$, provided E is a normed linear space.

We shall employ the following contractive definition: there exist $b \in [0, 1)$ and a monotone increasing function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\varphi(0) = 0$, such that for each $x, y \in E$,

$$d(Tx, Ty) \leq \varphi(d(x, Tx)) + bd(x, y). \tag{2.3}$$

The contractive definition (2.3) is more general than those considered by Berinde [1], Harder and Hicks [4], Rhoades [6, 7] and Osilike [8] in the following sense. If $\varphi(u) = Lu$ in (2.3) above, where $L \geq 0$ is a constant, then we obtain the contractive mapping of Osilike [8] which is itself a generalization of those in Berinde [1], Harder and Hicks [4] and Rhoades [7].

Again, if $L = mb$, $m = (1 - b)^{-1}, b \in [0, 1)$, we obtain the contractive condition considered by Rhoades [7]. Moreover, if $\varphi(u) = 0$, then (2.3) reduces to

$$d(Tx, Ty) \leq bd(x, y),\ b \in [0, 1),\ \forall x, y \in E,$$

which is a contractive definition used in Harder and Hicks [4] and Berinde[1]. Furthermore, if $L = 2\delta$, $b = \delta$, we obtain Zamfirescu contraction considered by Berinde [1], and Harder and Hicks [4], where

$$\delta = max\left\{\alpha, \frac{\beta}{1-\beta}, \frac{\gamma}{1-\gamma}\right\}, \tag{2.4}$$

$0 \leq \alpha < 1, 0 \leq \beta < 0.5$ and $0 \leq \gamma \leq 0.5$

The following Lemma of Berinde [1] shall be used in the proofs of the stability results.

Lemma 2.1. (Berinde [1]) *If* δ *is a real number such that* $0 \leq \delta < 1$*, and* $\{\epsilon_n\}_{n=0}^{\infty}$ *is a sequence of positive numbers such that* $\lim_{n\to\infty} \epsilon_n = 0$*, then for any sequence of positive numbers* $\{u_n\}_{n=0}^{\infty}$ *satisfying*

$$u_{n+1} \leq \delta u_n + \epsilon_n, n = 0, 1, ... \tag{2.5}$$

we have

$$\lim_{n\to\infty} u_n = 0.$$

Remark 1: The proof of this Lemma is contained in Berinde [1].

3. Main Results

The following is a stability result for the Picard iteration.

Theorem 3.1. *Let* (E, d) *be a complete metric space and* $T : E \to E$ *a selfmap of* E *satisfying* (2.3). *Suppose* T *has a fixed point* p^*. *Suppose also that* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *is monotone increasing and* $\varphi(0) = 0$. *Let* $x_0 \in E$ *and let* $x_{n+1} = Tx_n, n \geq 0$. *Then, Picard iteration process is T-stable.*

Proof. Let $\{y_n\}_{n=0}^{\infty} \subset E$ and $\epsilon_n = d(y_{n+1}, Ty_n)$.

Assume $\lim_{n\to\infty} \epsilon_n = 0$. Then, we shall establish that $\lim_{n\to\infty} y_n = p^*$ by using (2.3) and the triangle inequality :

$$\begin{aligned} d(y_{n+1}, p^*) &\leq d(y_{n+1}, Ty_n) + d(Ty_n, p^*) \\ &= \epsilon_n + d(Ty_n, Tp^*) \\ &= \epsilon_n + d(Tp^*, Ty_n) \\ &\leq \epsilon_n + \varphi(d(p^*, Tp^*)) + bd(p^*, y_n) \\ &= \epsilon_n + \varphi(d(p^*, p^*)) + bd(y_n, p^*) \\ &= \epsilon_n + \varphi(0) + bd(y_n, p^*) \\ &= bd(y_n, p^*) + \epsilon_n. \qquad (3.6) \end{aligned}$$

Using Lemma 1 in (3.6) yields $\lim_{n\to\infty} d(y_n, p^*) = 0$, that is,

$$\lim_{n\to\infty} y_n = p^*.$$

Conversely, suppose that $\lim_{n\to\infty} y_n = p^*$. Then,

$$\begin{aligned} \epsilon_n &= d(y_{n+1}, Ty_n) \leq d(y_{n+1}, p^*) + d(p^*, Ty_n) \\ &= d(y_{n+1}, p^*) + d(Tp^*, Ty_n) \\ &\leq d(y_{n+1}, p^*) + \varphi(d(p^*, Tp^*)) + bd(p^*, y_n) \\ &= d(y_{n+1}, p^*) + \varphi(d(p^*, p^*)) + bd(y_n, p^*) \\ &= d(y_{n+1}, p^*) + \varphi(0) + bd(y_n, p^*) \\ &= d(y_{n+1}, p^*) + bd(y_n, p^*) \to 0 \text{ as } n \to \infty. \end{aligned}$$

Remark 2: The Picard iteration is also T-stable for the more general contractive definition used in this paper than those employed in Berinde [1], Harder and Hicks [4], Rhoades [7] and Osilike [8]. If $\varphi(u) = Lu, L \geq 0$ is a constant, we obtain the contractive condition considered by Osilike [8] which is itself a generalization of Rhoades [7] and others earlier mentioned.

Theorem 1 in this paper is a generalization of Theorem 1 in Osilike [8]. We next prove a stability result for the Mann iteration process.

Theorem 3.2. *Let $(E, \|.\|)$ be a normed linear space and let $T : E \to E$ be a selfmap of E satisfying (2.3). Suppose T has a fixed point p^*. Let $x_0 \in E$ and suppose that $x_{n+1} = f(T, x_n) = (1-\alpha_n)x_n + \alpha_n Tx_n, n \geq 0,$ where $\{\alpha_n\}_{n=0}^{\infty}$ is a real sequence in [0,1] such that $0 < \alpha \leq \alpha_n$, $n = 0, 1, 2, \ldots.$ Suppose also that $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is monotone increasing and $\varphi(0) = 0$. Then, the Mann iteration process is T-stable.*

Proof. Let $\{y_n\}_{n=0}^{\infty} \subset E$ and define

$$\epsilon_n = \|y_{n+1} - (1-\alpha_n)y_n - \alpha_n Ty_n\|, n \geq 0.$$

Let $\lim_{n\to\infty} \epsilon_n = 0$. Then,

$$\begin{aligned}
\|y_{n+1} - p^*\| &\leq \|y_{n+1} - (1-\alpha_n)y_n - \alpha_n Ty_n\| + \|(1-\alpha_n)y_n + \alpha_n Ty_n - p^*\| \\
&= {}^{\epsilon}_{n} + \|(1-\alpha_n)y_n + \alpha_n Ty_n - ((1-\alpha_n) + \alpha_n)p^*\| \\
&= \epsilon_n + \|(1-\alpha_n)(y_n - p^*) + \alpha_n(Ty_n - p^*)\| \\
&\leq \epsilon_n + (1-\alpha_n)\|y_n - p^*\| + \alpha_n \|(Ty_n - p^*)\| \\
&= \epsilon_n + (1-\alpha_n)\|(y_n - p^*)\| + \alpha_n \|Ty_n - Tp^*\| \\
&= \epsilon_n + (1-\alpha_n)\|(y_n - p^*)\| + \alpha_n \|Tp^* - Ty_n\| \\
&\leq \epsilon_n + (1-\alpha_n)\|(y_n - p^*)\| + \alpha_n[\varphi(\|p^* - Tp^*\|) + b\|p^* - y_n\|] \\
&= (1-\alpha_n)\|(y_n - p^*)\| + b\alpha_n \|y_n - p^*\| + \alpha_n\varphi(\|p^* - p^*\|) + \epsilon_n \\
&= (1-\alpha_n + \alpha_n b)\|y_n - p^*)\| + \alpha_n\varphi(0) + \epsilon_n \\
&= (1-\alpha_n + \alpha_n b)\|y_n - p^*\| + \epsilon_n. \qquad (3.7)
\end{aligned}$$

Since $0 \leq 1 - \alpha_n + \alpha_n b < 1 - \alpha(1-b) < 1$, using Lemma 1 in (3.7) yields

$$\lim_{n\to\infty} \|y_n - p^*\| = 0,$$

that is,

$$\lim_{n\to\infty} y_n = p^*.$$

Conversely, let $\lim_{n\to\infty} y_n = p^*$. Then,

$$\begin{aligned}
\epsilon_n &= \|y_{n+1} - (1-\alpha_n)y_n - \alpha_n T y_n\| \\
&\leq \|(y_{n+1} - p^*\| + \|p^* - (1-\alpha_n)y_n - \alpha_n T y_n\| \\
&= \|y_{n+1} - p^*)\| + \|[(1-\alpha_n)+\alpha_n]p^* - (1-\alpha_n)y_n - \alpha_n T y_n\| \\
&= \|y_{n+1} - p^*\| + \|(1-\alpha_n)(p^* - y_n) + \alpha_n(p^* - Ty_n)\| \\
&\leq \|y_{n+1} - p^*\| + (1-\alpha_n)\|y_n - p^*\| + \alpha_n\|p^* - Ty_n \\
&= \|y_{n+1} - p^*\| + (1-\alpha_n)\|y_n - p^*\| + \alpha_n\|Tp^* - Ty_n\| \\
&\leq \|y_{n+1} - p^*\| + (1-\alpha_n)\|y_n - p^*\| + \alpha_n[\varphi(\|p^* - Tp^*\|) + b\|p^* - y_n\|] \\
&= \|y_{n+1} - p^*\| + (1-\alpha_n)\|y_n - p^*\| + \alpha_n b\|y_n - p^*\| + \alpha_n\varphi(\|p^* - p^*\|) \\
&= \|y_{n+1} - p^*\| + (1-\alpha_n+\alpha_n b)\|y_n - p^*\| + \alpha_n\varphi(0) \\
&= \|y_{n+1} - p^*\| + (1-\alpha_n+\alpha_n b)\|y_n - p^*\| \to 0 \text{ as } n\to\infty.
\end{aligned}$$

Remark 3: The Mann iteration process is T-stable for a more general contractive definition than those in Berinde [1], Osilike [8], Rhoades [7], Harder and Hicks [4]. Theorem 2 in this paper is a generalization of Theorem 2 in Rhoades [7].

Acknowledgement.

The second author thanks Professor V. Berinde for supplying the reprints of his works which were very useful in the preparation of this paper.

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