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A survey on the approximation properties of Schurer-Stancu operators

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ABSTRACT. The paper is a survey of some results obtained by the author in the last two years, concerning approximation properties of Schurer-Stancu operators.

1. PRELIMINARIES

Let $p \in \mathbb{N}$ be a given integer and let $\alpha, \beta \in \mathbb{R}$ be given parameters satisfying the conditions $0 \leq \alpha \leq \beta$.

The Schurer-Stancu operators $\widetilde{S}_{m,p}^{(\alpha,\beta)} : C([0,1+p]) \to C([0,1])$ are defined for any $m \in \mathbb{N}$ and any $f \in C([0,1+p])$ by

$$(1.1) \qquad \left(\widetilde{S}_{m,p}^{(\alpha,\beta)} f\right)(x) = \sum_{k=0}^{m+p} \widetilde{p}_{m,n}(x) f\left(\frac{k+\alpha}{m+p}\right)$$

where

$$(1.2) \qquad \widetilde{p}_{m,n}(x) = \binom{m+p}{k} x^k (1-x)^{m+p-k}$$

are the fundamental Schurer polynomials (see [4]).

Note that the operators (1.1) belong to a class of more general linear operators introduced in 1996 by Stancu, D.D. (see [19]).

For $\alpha = \beta = 0$, the operators (1.1) are the Schurer operators $\widetilde{B}_{m,p} : C([0,1+p]) \to C([0,1+p])$, defined for any $n \in \mathbb{N}$ and any $f \in C([0,1+p])$ by

$$(1.3) \qquad \left(\widetilde{B}_{m,p} f\right)(x) = \sum_{k=0}^{m+p} \widetilde{p}_{m,k}(x) f\left(\frac{k}{m}\right)$$

(see [16]).

For $p = 0$, the operators (1.1) are the Stancu operators $\widetilde{P}_m^{(\alpha,\beta)} : C([0,1]) \to C([0,1])$ defined for any $m \in \mathbb{N}$ and any $f \in C([0,1])$ by

$$(1.4) \qquad \left(\widetilde{P}_m^{(\alpha,\beta)} f\right)(x) = \sum_{k=0}^{m} p_{m,k}(x) f\left(\frac{k+\alpha}{m+\beta}\right)$$

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where

$$(1.5)\qquad p_{m,k}(x)=\binom{m}{k}x^k(1-x)^{m-k}$$

are the fundamental Bernstein polynomials (see [1]).

For $p=0$ and $\alpha=\beta=0$, the operators (1.1) are the classical Bernstein operators $B_m : C([0,1])\to C([0,1])$, defined for any $m\in\mathbb{N}$ and any $f\in C([0,1])$ by

$$(1.6)\qquad (B_m f)(x)=\sum_{k=0}^{m}p_{m,k}(x)f\left(\frac{k}{m}\right)$$

(see [14]).

2. Results

Let $\widetilde{S}_{m,p}^{(\alpha,\beta)}$ be the Schurer-Stancu operators (1.1).

Theorem 2.1. [4]. *The sequence* $\left\{\widetilde{S}_{m,p}^{(\alpha,\beta)}\right\}_{m\in\mathbb{N}}$ *converges to* f*, uniformly on* $[0,1]$*, for any* $f\in C([0,1+p])$.

As consequences of Theorem 2.1 one obtain:

Corollary 2.1. [16]. *Let* $\alpha=\beta=0$ *and let* $\widetilde{B}_{m,p}=\widetilde{S}_{m,p}^{(0,0)}$ *be the Schurer operator* (1.3)*. The sequence* $\left\{\widetilde{B}_{m,p}f\right\}_{m\in\mathbb{N}}$ *converges to* f*, uniformly on* $[0,1]$*, for any* $f\in C([0,1+p])$.

Corollary 2.2. [17]. *Let* $p=0$ *and let* $\widetilde{P}_{m}^{(\alpha,\beta)}=\widetilde{S}_{m,0}^{(\alpha,\beta)}$ *be the Stancu operator* (1.4)*. The sequence* $\left\{P_{m}^{(\alpha,\beta)}f\right\}_{m\in\mathbb{N}}$ *converges to* f*, uniformly on* $[0,1]$*, for any* $f\in C([0,1])$.

Corollary 2.3. [14]. *Let* $p=\alpha=\beta=0$ *and let* $B_m=\widetilde{S}_{m,0}^{(0,0)}$ *be the classical Bernstein operator* (1.6)*. The sequence* $\{B_m f\}_{m\in\mathbb{N}}$ *converges to* f*, uniformly on* $[0,1]$*, for any* $f\in C([0,1])$.

Let $\omega_1 : [0,+\infty)\to\mathbb{R}$ be the first order modulus of smoothness (or modulus of continuity). The order of local approximation of $f\in C([0,1+p])$ by $\widetilde{S}_{m,0}^{(\alpha,\beta)}f$ is contained in

Theorem 2.2. [4]. *For any* $f\in C([0,1+p])$ *and each* $x\in[0,1]$*, the following*

$$(2.7)\qquad \left|\left(\widetilde{S}_{m,p}^{(\alpha,\beta)}f\right)(x)-f(x)\right|\le 2\omega_1\left(\sqrt{\delta_{m,p,\alpha,\beta,x}}\right)$$

holds, where

$$(2.8)\qquad \begin{aligned}\delta_{m,p,\alpha,\beta,x} &= \frac{(p-\beta)^2}{(m+\beta)^2}x^2+\frac{m+p}{(m+\beta)^2}\,x(1-x)+\\ &+ \frac{2\alpha(mp-2m\beta-\beta^2)}{(m+\beta)^2}\,x+\frac{\alpha^2(3m+\beta)}{(m+\beta)^2}\,.\end{aligned}$$

Taking the maximum in (2.8) yields the order of global approximation of f by $\widetilde{S}_{m,p}^{(\alpha\beta)}f$.

As particular cases, from Theorem 2.2 one obtain the orders of local approximation by Schurer, Stancu and respectively Bernstein operators.

The next theorem contains a nice result related to simultaneous approximation by the operators (1.1).

Theorem 2.3. [9]. *Let j be a positive integer and let D^j be the $j-th$ order differential operator. For any $f \in C^j([0,1+p])$ the sequence $\left\{D^j\left(\widetilde{S}_{m,p}^{(\alpha,\beta)}f\right)\right\}_{m\in\mathbb{N}}$ converges to D^jf, uniformly on $[0,1]$.*

A consequences of Theorem 2.3 one obtain:

Corollary 2.4. [8]. *The sequence $\left\{D^j\left(\widetilde{B}_{m,p}f\right)\right\}_{m\in\mathbb{N}}$ converges to D^jf, uniformly on $[0,1]$, for any $f \in C^j([0,1+p])$.*

Corollary 2.5. [2]. *The sequence $\left\{D^j\widetilde{P}_m^{(\alpha,\beta)}f\right\}_{m\in\mathbb{N}}$ converges to D^jf, uniformly on $[0,1]$, for any $f \in C^j([0,1])$.*

Corollary 2.6. [1]. *The sequence $\left\{D^jB_mf\right\}_{m\in\mathbb{N}}$ converges to D^jf, uniformly on $[0,1]$, for any $f \in C^j([0,1])$.*

Following the ideas of Kantorovich, L.V. [15], in [11] we constructed the operators $\widetilde{K}_{m,p}^{(\alpha,\beta)} : L_1([0,1]) \to C([0,1])$ defined for any $f \in L_1([0,1])$ and any $m \in \mathbb{N}$ by

$$\left(K_{m,p}^{(\alpha,\beta)}f\right)(x) = (m+p+\beta+1)\sum_{k=0}^{m+p}\widetilde{p}_{m,k}(x)\int_{\frac{k+\alpha}{m+p+\beta+1}}^{\frac{k+\alpha+1}{m+p+\beta+1}} f(s)ds \tag{2.9}$$

Theorem 2.4. [11]. *The sequence $\left\{\widetilde{K}_{m,p}^{(\alpha,\beta)}f\right\}_{m\in\mathbb{N}}$ converges to f, uniformly on $[0,1]$, for any $f \in L_1([0,1])$.*

Theorem 2.5. [11]. *For any $f \in L_1([0,1])$ and each $x \in [0,1]$, the following inequality*

$$\left|\left(\widetilde{K}_{m,p}^{(\alpha,\beta)}f\right)(x) - f(x)\right| \le 2\omega_1\left(f;\sqrt{\delta_{m,p}^{(\alpha,\beta)}(x)}\right) \tag{2.10}$$

holds, where

$$\delta_{m,p}^{(\alpha,\beta)}(x) = \widetilde{K}_{m,p}^{(\alpha,\beta)}\left((e_1-x)^2;x\right) \tag{2.11}$$

Theorem 2.6. [11]. *For any* $f \in C^1([0,1])$ *and each* $x \in [0,1]$, *the following inequality*

$$\left|\left(\widetilde{K}_{m,p}^{(\alpha,\beta)} f\right)(x) - f(x)\right| \leq \tag{2.12}$$

$$\leq |f'(x)| \left| \frac{m+\beta}{2(m+p+\beta+1)^2} - \frac{\beta+1}{(m+p+\beta+1)^2}\, x \right| +$$

$$+ 2\sqrt{\delta_{m,p}^{(\alpha,\beta)}(x)}\, \omega_1 \left(f'; \sqrt{\delta_{m,p}^{(\alpha,\beta)}(x)}\right)$$

holds.

Remark 2.1. For $p = 0$, $\widetilde{K}_{m,p}^{(\alpha,\beta)} = K_{m,p}^{(\alpha,\beta)}$ are the so called Kantorovich-Stancu operators (see [10], [11]). Some of them approximation properties can be obtained from Theorem 2.4, Theorem 2.5 and Theorem 2.6.

Remark 2.2. For $\alpha = \beta = 0$, $\widetilde{K}_{m,0}^{(0,0)} = \widetilde{K}_{m,p}$ are the Kantorovich-Schurer operators (see [10]) and their approximation properties follow from Theorem 2.4, Theorem 2.5 and Theorem 2.6.

Remark 2.3. For $\alpha = \beta = p = 0$, $\widetilde{K}_{m,0}^{(0,0)} = K_m$ are the classical Kantorovich operators (see [15]).

Remark 2.4. In our earlier papers [5], [6], [7], [8], [12] were also introduced and studied extensions of operators $\widetilde{S}_{m,p}^{(\alpha,\beta)}$ and $\widetilde{K}_{m,p}^{(\alpha,\beta)}$ to the case of bivariate functions.

Using the method of parametric extensions [3], in [6] we introduced the bivariate Schurer-Stancu operators $\widetilde{S}_{m,n,pq}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)} : C\big([0,1+p]\times[0,1+q]\big) \to C\big([0,1]\times[0,1]\big)$, defined by

$$\left(\widetilde{S}_{m,n,p,q}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)} f\right)(x,y) = \sum_{k=0}^{m+p} \sum_{j=0}^{n+q} \widetilde{p}_{m,k}(x)\widetilde{p}_{n,j}(y) f\left(\frac{k+\alpha_1}{m+\beta_1}, \frac{j+\alpha_2}{n+\beta_2}\right) \tag{2.13}$$

where $\widetilde{p}_{m,k}(x)$, $\widetilde{p}_{n,j}(y)$ are the fundamental Schurer polynomials (1.2), p, q are non-negative integers and α_1, β_1, α_2, β_2 are real parameters satisfying the conditions $0 \leq \alpha_1 \leq \beta_1$, $0 \leq \alpha_2 \leq \beta_2$.

We proved

Theorem 2.7. [6]. *For any* $f \in C\big([0,1+p]\times[0,1+q]\big)$ *the sequence*

$$\left\{\widetilde{S}_{m,n,p,q}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)} f\right\}_{m,n\in\mathbb{N}}$$

converges to f *uniformly on* $[0,1]\times[0,1]$.

The approximation order of f by $\widetilde{S}_{m,n,p,q}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)} f$ in terms of first order modulus of smoothness for bivariate functions was also established in [6].

In [5] we constructed the GBS operator of Schurer-stancu type and we studied some of them approximation properties were studied.

Some integral forms of operators $\widetilde{S}_{m,n,p,q}^{(\alpha_1,\beta_1,\alpha_2,\beta_2)}$ were introduced and studied in [7] and [10].

Now, we are dealing with other approximation properties of Schurer-Stancu operators, which, we hope, will be published in the future.

References

[1] Agratini, O., *Aproximare prin operatori liniari*, Presa Universitară Clujeană, Cluj-Napoca, 2000

[2] Agratini, O., *On simultaneous approximation by Stancu-Bernstein operators*, Proceed of I.C.A.O.R: Approximation and Optimization Press, Cluj-Napoca, **I** (1977), 157-162

[3] Bărbosu, D., *Aproximarea funcţiilor de mai multe variabile prin sume booleene de operatori liniari de tip interpolator*, Ed. Risoprint, Cluj-Napoca, 2003

[4] Bărbosu, D., *Schurer-Stancu type operators*, Studia Univ. "Babeş-Bolyai", Mathematica, **XLVIII** (2003), No. 3, 31-35

[5] Bărbosu, D., *GBS operators of Schurer-Stancu type*, Anals of University of Craiova, Math. Comp. Science, **31** (2003), 1-7

[6] Bărbosu, D., *Bivariate operators of Schurer-Stancu type*, Anal. Univ. "Ovidius" Constanţa, Ser. Math. **XI** (2003), Fasc. 1, 1-8

[7] Bărbosu, D., Bărbosu, M., *GBS operators of Kantorovich-Schurer type*, Bulletins for Applied &Computer Mathematics, BAM-2076, A(CIV), Budapest (2003), 339-346

[8] Bărbosu, D., Bărbosu, M., *Simultaneous approximation by Schurer type operators*, Carpathian J. Math., **19** (2003), 1-6

[9] Bărbosu, D., *Simultaneous approximation by Schurer-Stancu type operators*, Math. Balkanica, **17** (2003), Fasc. 3-4, 363-372

[10] Bărbosu, D., *Kantorovich-Stancu type operators*, JIPAM (Journal of Inequalities in Pure and Applied Mathematics), **5** (2004), No. 3, 1-6

[11] Bărbosu, D., *The Kantorovich form of Schurer-Stancu operators*, Dem. Mathematica, **XXXVII** (2004), No. 2, 383-391

[12] Bărbosu, D., *Kantorovich-Schurer bivariate operators*, Miskolc, Math. Notes, **5** (2004), No. 2, 129-136

[13] Bărbosu, D., *Simultaneous approximation by some Kantorovich type operators*, (to appear in Dem. Math. **XXXVIII** (2005))

[14] Bernstein, S.N., *Démonstration du théorème de Weierstrass fondée sur le calcul des probabilités Commun*, Soc. Math. Kharkow (2), **13** (1912-13), 1-2

[15] Kantorovich, L.V., *Sur certain développements suivant les pôlynômes de la forme de S. Bernstein*, I, II, C.R. Acad. URSS (1930), 563-568, 595-600

[16] Schurer, F., *Linear positive operators in approximation theory*, Math. Inst. Tech. Univ. Delft: Report, 1962

[17] Stancu, D.D., *Approximation of function by a new class of linear polynomial operators*, Rev. Roum. Math. Pures et Appl., **13** (1968), No. 8, 1173-1194

[18] Stancu, D.D., *Asupra unei generalizări a polinoamelor lui Bernstein*, Studia Univ. "Bebeş-Bolyai", **14** (1969), No. 2, 31-42

[19] Stancu, D.D., *Approximation properties of a class of multiparameter positive linear operators*, Approximation and Optimization Transilvania Press, Cluj-Napoca, **I** (1977), 107-120

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