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The connection between the implicit function theorem and the existence theorem for differential equations

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ABSTRACT. The paper presents a variant of implicit function theorem, as a corollary of existence theorem for solutions of a first order system of partial differential equations.

This results is then extend to the case of implicit function theorem for functions of several independent variables.

1. INTRODUCTION

The aim of this paper is to show how a theorem on the existence of solutions to a first order system of partial differential equations can be used to prove the implicit function theorem.

For a system of ordinary differential equations, in [5, p. 62], it is proved that the classical Picard existence theorem, implies, as a corollary, the implicit function theorem.

First, we consider the first order system of partial differential equation

$$(1.1)\qquad \begin{cases} \dfrac{\partial z}{\partial x} &= F_1(x,y,z(x,y)), \\ \dfrac{\partial z}{\partial y} &= F_2(x,y,z(x,y)), \end{cases} ;\ (x,y)\in \overline{D},$$

where $\overline{D} = [x_0 - a, x_0 + a] \times [y_0 - b, y_0 + b]$, $a > 0$, $b > 0$, with initial condition

$$(1.2)\qquad z(x_0, y_0) = z_0,$$

where $(x_0, y_0, z_0) \in \Omega$, $\Omega = \overline{D} \times \mathbb{R}$, $F_1, F_2 : \Omega \to \mathbb{R}$, are given.

On the existence of the solutions of the problem (1.1) + (1.2), in [3, p. 423] and [7, p. 219] is proved a result given by:

Theorem 1.1. *Suppose that $F_1, F_2 \in C^1(\Omega)$ and $\dfrac{\partial F_1}{\partial z}$, $\dfrac{\partial F_2}{\partial z}$ are bounded. If*

$$(1.3)\qquad \frac{\partial F_1}{\partial y} + \frac{\partial F_1}{\partial z}\cdot F_2 = \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial z}\cdot F_1,\ (x,y,z)\in\Omega$$

(*the complete integrability condition of the system* (1.1)), *then there exists an unique solution $z(x,y)$, $z \in C^1(D)$, of the problem* (1.1) + (1.2).

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2. The main result

Let the equation

$$H(x,y,z)=0, \tag{2.4}$$

be, where $H:\Omega\to\mathbb{R}$ is $C^1(\Omega)$, $\Omega=\overline{D}\times\mathbb{R}$, $\overline{D}=[x_0-a,x_0+a]\times[y_0-b,y_0+b]$, $a>0$, $b>0$.

Using Theorem 1.1, we can prove the implicit function theorem, defined by the equation (2.4) (see [2, p. 401]).

Theorem 2.2. *Suppose that the hypothesises of the Theorem 1.1 hold. If for point $(x_0,y_0,z_0)\in\Omega$ we have*

$$H(x_0,y_0,z_0)=0, \tag{2.5}$$

and

$$\frac{\partial H}{\partial z}(x_0,y_0,z_0)\neq 0, \tag{2.6}$$

then, there exists an unique implicit function $\Phi:D\to\mathbb{R}$, $\Phi\in C^1(D)$ such that

$$\Phi(x_0,y_0)=z_0, \tag{2.7}$$

$$H(x,y,\Phi(x,y))=0,\quad (x,y)\in D, \tag{2.8}$$

and

$$\frac{\partial\Phi}{\partial x}=-\frac{\dfrac{\partial H}{\partial x}}{\dfrac{\partial H}{\partial z}};\ \frac{\partial\Phi}{\partial y}=-\frac{\dfrac{\partial H}{\partial y}}{\dfrac{\partial H}{\partial z}}. \tag{2.9}$$

Proof. First, choose $a,b,r>0$ so that

$$\Delta=(x_0-a,x_0+a)\times(y_0-b,y_0+b)\times(z_0-r,z_0+r)\subseteq\Omega,$$

and $\dfrac{\partial H}{\partial z}(x,y,z)$ is non vanishing on Δ. Then define $F:\Delta\to\mathbb{R}^2$, $F=(F_1,F_2)$ by setting

$$F_1(x,y,z)=-\frac{\dfrac{\partial H}{\partial x}(x,y,z)}{\dfrac{\partial H}{\partial z}(x,y,z)}, \tag{2.10}$$

$$F_2(x,y,z)=-\frac{\dfrac{\partial H}{\partial y}(x,y,z)}{\dfrac{\partial H}{\partial z}(x,y,z)}. \tag{2.11}$$

Since the hypothesises of Theorem 1.1 are satisfied, we can apply this theorem to conclude that there exists a unique solution $z(x,y)$ of the problem

$$\begin{cases}\dfrac{\partial z}{\partial x} &= F_1(x,y,z(x,y)),\\[2mm] \dfrac{\partial z}{\partial y} &= F_2(x,y,z(x,y)),\end{cases}\qquad z(x_0,y_0)=z_0$$

defined on the interval $D = (x_0 - a, x_0 + a) \times (y_0 - b, y_0 + b)$.

Let $\Phi : (x_0 - a, x_0 + a) \times (y_0 - b, y_0 + b) \to \mathbb{R}$,

$$\Phi(x, y) = z(x, y), \quad (x, y) \in D$$

We note that

$$\Phi(x_0, y_0) = z(x_0, y_0) = z_0, \tag{2.12}$$

and using (2.10), (2.11)

$$\begin{cases} \dfrac{\partial \Phi}{\partial x} = \dfrac{\partial z}{\partial x} = F_1(x, y, z(x, y)) = -\dfrac{\dfrac{\partial H}{\partial x}(x, y, \Phi(x, y))}{\dfrac{\partial H}{\partial z}(x, y, \Phi(x, y))}, \\ \\ \dfrac{\partial \Phi}{\partial y} = \dfrac{\partial z}{\partial y} = F_2(x, y, z(x, y)) = -\dfrac{\dfrac{\partial H}{\partial y}(x, y, \Phi(x, y))}{\dfrac{\partial H}{\partial z}(x, y, \Phi(x, y))}. \end{cases} \tag{2.13}$$

By (2.12), (2.5), we have $H(x_0, y_0, \Phi(x_0, y_0)) = H(x_0, y_0, z_0) = 0$, and by (2.13), we have

$$\begin{aligned} &\frac{\partial}{\partial x} H(x, y, \Phi(x, y)) = \\ &= \frac{\partial H}{\partial x}(x, y, \Phi(x, y)) + \frac{\partial H}{\partial z}(x, y, \Phi(x, y)) \frac{\partial \Phi}{\partial x} = \frac{\partial H}{\partial x}(x, y, \Phi(x, y)) - \\ &- \frac{\dfrac{\partial H}{\partial z}(x, y, \Phi(x, y)) \dfrac{\partial H}{\partial x}(x, y, \Phi(x, y))}{\dfrac{\partial H}{\partial z}(x, y, \Phi(x, y))} = 0. \end{aligned}$$

Analogous

$$\frac{\partial}{\partial y} H(x, y, \Phi(x, y)) = 0, \quad (x, y) \in D.$$

Thus we obtain $H(x, y, \Phi(x, y)) = c = \text{constant} = 0$, $(x, y) \in D$, because of $H(x_0, y_0, \Phi(x_0, y_0)) = H(x_0, y_0, z_0) = 0$.

The proof is now complete. □

3. The case of several independent variables

We can extend now the application of Theorem 1.1 and Theorem 2.1 in case of several independent variables.

Consider the system of nonlinear partial differential equations

$$\begin{cases} \dfrac{\partial y}{\partial x_1} = F_1(x_1, x_2, \ldots, x_m, y(x_1, x_2, \ldots, x_m)), \\ \dfrac{\partial y}{\partial x_2} = F_2(x_1, x_2, \ldots, x_m, y(x_1, x_2, \ldots, x_m)), \\ \cdot \;\cdot \;\cdot \;\cdot \;\cdot \;\cdot \;\cdot \;\cdot \;\cdot \;\cdot \;\cdot \;\cdot \;\cdot \;\cdot \;\cdot \\ \dfrac{\partial y}{\partial x_m} = F_m(x_1, x_2, \ldots, x_m, y(x_1, x_2, \ldots, x_m)), \end{cases} \tag{3.14}$$

with initial condition

(3.15) $\quad y(x_1^0, x_2^0, \dots, x_m^0) = y^0\,.$

Here $F_i : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}^{m+1}$, $\Omega = \overline{D} \times \mathbb{R}$,

$$\overline{D} = \left[x_1^0 - a_1, x_1^0 + a_1\right] \times \left[x_2^0 - a_2, x_2^0 + a_2\right] \times \dots \times \left[x_m^0 - a_m, x_m^0 + a_m\right],$$

$a_i > 0$, $i = 1, 2, \dots, m$, $\left(x_1^0, x_2^0, \dots, x_m^0, y^0\right) \in \Omega$, are given.

For initial value problem (3.14) + (3.15) we have

Theorem 3.3. *Suppose that $F_1, F_2, \dots, F_m \in C^1(\Omega)$ and $\dfrac{\partial F_i}{\partial y}$, $i = 1, 2, \dots, m$ are bounded. If the complete integrability conditions*

(3.16) $$\frac{\partial F_i}{\partial x_j} + \frac{\partial F_i}{\partial y}\,F_j = \frac{\partial F_j}{\partial x_i} + \frac{\partial F_j}{\partial y}\,F_i,$$

are satisfied for all $i, j \in \{1, 2, \dots, m\}$, $i \neq j$, $(x_1, x_2, \dots, x_m, y) \in \Omega$, then, there exists an unique solution $y(x_1, x_2, \dots, x_m)$, $y \in C^1(D)$, of the problem (3.14) + (3.15).

Proof. It is not a difficult problem to show that the initial value problem (3.14) + (3.15) is equivalent to the Volterra integral equation

(3.17) $$\begin{aligned} y(x_1, x_2, \dots, x_m) &= y^0 + \int_{x_1^0}^{x_1} F_1\big(s_1, x_2^0, \dots, x_m^0, y(s_1, x_2^0, \dots, x_m^0)\big)ds_1 + \\ &+ \int_{x_2^0}^{x_2} F_2\big(x_1, s_2, x_3^0, \dots, x_m^0, y(x_1, s_2, x_3^0, \dots, x_m^0)\big)ds_2 + \dots + \\ &+ \int_{x_{m-1}^0}^{x_{m-1}} F_{m-1}\big(x_1, x_2, \dots, x_{m-2}, s_{m-1}, x_m^0, y(x_1, \dots, x_{m-1}, s_{m-1}, x_m^0)\big)ds_{m-1} + \\ &+ \int_{x_m^0}^{x_m} F_m(x_1, x_2, \dots, x_{m-1}, s_m, y(x_1, x_2, \dots, x_{m-1}, s_m)\big)ds_m\,. \end{aligned}$$

Let the map $T : C(\overline{D}) \to C(\overline{D})$, be defined by the right side of (3.17), that is

(3.18) $$\begin{aligned} T(y)(x_1, x_2, \dots, x_m) &= \\ &= y^0 + \int_{x_1^0}^{x_1} F_1\big(s_1, x_2^0, \dots, x_m^0, y(s_1, x_2^0, \dots, x_m^0)\big)ds_1 + \dots + \\ &+ \int_{x_m^0}^{x_m} F_m\big(x_1, x_2, \dots, x_{m-1}, s_m, y(x_1, x_2, \dots, x_{m-1}, s_m)\big)ds_m\,. \end{aligned}$$

On the linear space $C(\overline{D})$, we define the Bielecki norm

(3.19) $$\|y\|_B := \max_{(x_1, x_2, \dots, x_m) \in \overline{D}} \big|y(x_1, x_2, \dots, x_m)\big| e^{-\tau\left(|x_1 - x_1^0| + |x_2 - x_2^0| + \dots + |x_m - x_m^0|\right)},$$

$y \in C(\overline{D})$, $\tau > 0$.

Now $\big(C(\overline{D}), \|\cdot\|\big)$ is a Banach space and with metric

$$\rho(y,z) := \|y-z\|_B; \quad y, z \in C(\overline{D}),$$

$\big(C(\overline{D}), \rho\big)$ is a complete metric space.

We can choose $\tau > 0$ so that the map T be a contraction on $C(\overline{D})$, that is

$$\rho\big(T(y), T(z)\big) \le \alpha\rho(y,z),\ y, z \in C(\overline{D}),\ 0 \le \alpha < 1.$$

We can apply Banach's theorem to conclude that the map T has an unique fix point, $y(x_1, x_2, \ldots, x_m) \in C(\overline{D})$ so that

$$y = Ty.$$

Then, the integral equation (3.17) has an unique solution $y(x_1, x_2, \ldots, x_m)$ and $y(x_1, x_2, \ldots, x_m)$ is also, the unique solution of the initial value problem (3.14) + (3.15). □

Remark 3.1. The complete integrability conditions (3.16) are necessary conditions for existence at least one solution of system (3.14) because if $y(x) = y(x_1, x_2, \ldots, x_m)$ is a C^2 solution of (3.14), then

$$\frac{\partial^2 y}{\partial x_i \partial x_j} = \frac{\partial^2 y}{\partial x_j \partial x_i}\,,$$

equivalent to

$$\frac{\partial F_i}{\partial x_j} + \frac{\partial F_i}{\partial y}\,\frac{\partial y}{\partial x_j} = \frac{\partial F_j}{\partial x_i} + \frac{\partial F_j}{\partial y}\,\frac{\partial y}{\partial x_i}\,.$$

This is just the condition (3.16).

We can now state an analogous of Theorem 2.1.

Let the equation

$$H(x_1, x_2, \ldots, x_m, y) = 0, \tag{3.20}$$

be, where $H : \Omega \to \mathbb{R}$ is a $C^1(\Omega)$, $\Omega = \overline{D} \times \mathbb{R}$,

$$\overline{D} = \big[x_1^0 - a_1, x_1^0 + a_1\big] \times \cdots \times \big[x_m^0 - a_m, x_m^0 + a_m\big],\ a_i > 0,\ i = \overline{1,m}.$$

Theorem 3.4. *Suppose that the hypothesises of the Theorem* 3.1 *hold. If the point* $(x_1^0, x_2^0, \ldots, x_m^0, y^0) \in \Omega$ *satisfy*

$$H(x_1^0, x_2^0, \ldots, x_m^0, y^0) = 0, \tag{3.21}$$

and

$$\frac{\partial H}{\partial y}\,(x_1^0, x_2^0, \ldots, x_m^0, y^0) \neq 0, \tag{3.22}$$

then, there exists an unique implicit function $\Phi : D \to \mathbb{R}$, $\Phi \in C^1(D)$ *such that*

$$\Phi(x_1^0, x_2^0, \ldots, x_m^0) = y^0, \tag{3.23}$$

$$H\big(x_1, x_2, \ldots, x_m, \Phi(x_1, x_2, \ldots, x_m)\big) = 0,\ (x_1, x_2, \ldots, x_m) \in D, \tag{3.24}$$

and for $i = 1, 2, \dots, m$ *and* $(x_1, x_2, \dots, x_m) \in D$

$$\frac{\partial \Phi}{\partial x_i} = -\frac{\dfrac{\partial H}{\partial x_i}\big(x_1, x_2, \dots, x_m, \Phi(x_1, x_2, \dots, x_m)\big)}{\dfrac{\partial H}{\partial y}\big(x_1, x_2, \dots, x_m, \Phi(x_1, \dots, x_m)\big)}. \tag{3.25}$$

Proof. The proof follows the line of proof of Theorem 2.1. Choose $a_1, a_2, \dots, a_m$, $r > 0$ so that

$$\begin{aligned}\Delta &= \big(x_1^0 - a_1, x_2^0 + a_1\big) \times \big(x_2^0 - a_2, x_2^0 + a_2\big) \times \dots \\ &\quad \times \big(x_m^0 - a_m, x_m^0 + a_m\big) \times \big(y^0 - r, y^0 + r\big) \subseteq \Omega,\end{aligned}$$

and $\dfrac{\partial H}{\partial y}(x_1, x_2, \dots, x_m, y)$ is non vanishing on Δ. This is possible because of continuity of $\dfrac{\partial H}{\partial y}$ and because of (3.22).

Then, define $F : \Delta \to \mathbb{R}^m$, $F = (F_1, F_2, \dots, F_m)$, with

$$\begin{aligned} F_1(x_1, y_2, \dots, x_m, y) &= -\frac{\dfrac{\partial H}{\partial x_1}(x_1, x_2, \dots, x_m, y)}{\dfrac{\partial H}{\partial y}(x_1, x_2, \dots, x_m, y)}, \\ F_2(x_1, x_2, \dots, x_m, y) &= -\frac{\dfrac{\partial H}{\partial x_2}(x_1, x_2, \dots, x_m, y)}{\dfrac{\partial H}{\partial y}(x_1, x_2, \dots, x_m, y)}, \\ &\dots\dots\dots\dots\dots\dots \\ F_m(x_1, x_2, \dots, x_m, y) &= -\frac{\dfrac{\partial H}{\partial x_m}(x_1, x_2, \dots, x_m, y)}{\dfrac{\partial H}{\partial y}(x_1, x_2, \dots, x_m, y)}. \end{aligned} \tag{3.26}$$

Since the hypothesises of Theorem 3.1 are satisfied, we can apply this theorem to conclude that there exists a unique solution $y(x_1, x_2, \dots, x_m)$ of the initial value problem

$$\begin{cases} \dfrac{\partial y}{\partial x_1} = F_1\big(x_1, x_2, \dots, x_m, y(x_1, x_2, \dots, x_m)\big), \\ \dfrac{\partial y}{\partial x_2} = F_2\big(x_1, x_2, \dots, x_m, y(x_1, x_2, \dots, x_m)\big), \\ \dots\dots\dots\dots\dots \\ \dfrac{\partial y}{\partial x_m} = F_2\big(x_1, x_2, \dots, x_m, y(x_1, x_2, \dots, x_m)\big), \end{cases} \quad y(x_1^0, x_2^0, \dots, x_m^0) = y^0, \tag{3.27}$$

where F_i are defined by (3.26). Define

$$\Phi : (x_1^0 - a_1, x_1^0 + a_1) \times \cdots \times (x_m^0 - a_m, x_m^0 + a_m) \to \mathbb{R},$$

$$\Phi(x_1, x_2, \ldots, x_m) = y(x_1, x_2, \ldots, x_m),\ (x_1, x_2, \ldots, x_m) \in D.$$

We observe that

$$\Phi(x_1^0, x_2^0, \ldots, x_m^0) = y(x_1^0, x_2^0, \ldots, x_m^0) = y^0, \tag{3.28}$$

and using (3.26) we have

$$\begin{aligned} \frac{\partial \Phi}{\partial x_i} = \frac{\partial y}{\partial x_i} &= F_i\big(x_1, x_2, \ldots, x_m, y(x_1, x_2, \ldots, x_m)\big) = \\ &= F_i\big(x_1, x_2, \ldots, x_m, \Phi(x_1, x_2, \ldots, x_m)\big) = \\ &= -\frac{\dfrac{\partial H}{\partial x_i}\big(x_1, x_2, \ldots, x_m, \Phi(x_1, x_2, \ldots, x_m)\big)}{\dfrac{\partial H}{\partial y}\big(x_1, x_2, \ldots, x_m, \Phi(x_1, x_2, \ldots, x_m)\big)}, \end{aligned} \tag{3.29}$$

that is $\Phi \in C^1(D)$ and (3.25) are satisfied.

Moreover, from (3.21) and (3.28) we get

$$H\big(x_1^0, x_2^0, \ldots, x_m^0, \Phi(x_1^0, x_2^0, \ldots, x_m^0)\big) = H(x_1^0, x_2^0, \ldots, x_m^0, y^0) = 0, \tag{3.30}$$

and for $i = 1, 2, \ldots, m$, using (3.29)

$$\begin{aligned} &\frac{\partial}{\partial x_i} H\big(x_1, x_2, \ldots, x_m, \Phi(x_1, x_2, \ldots, x_m)\big) = \frac{\partial H}{\partial x_i} + \frac{\partial H}{\partial y} \cdot \frac{\partial \Phi}{\partial x_i} = \\ &= \frac{\partial H}{\partial x_i}\big(x_1, x_2, \ldots, x_m, \Phi(x_1, x_2, \ldots, x_m)\big) - \\ &- \frac{\partial H}{\partial y}\big(x_1, x_2, \ldots, x_m, \Phi(x_1, x_2, \ldots, x_m)\big) \cdot \\ &\cdot \frac{\dfrac{\partial H}{\partial x_i}\big(x_1, x_2, \ldots, x_m, \Phi(x_1, \ldots, x_m)\big)}{\dfrac{\partial H}{\partial y}\big(x_1, \ldots, x_m, \Phi(x_1, \ldots, x_m)\big)} = 0,\ \forall\, (x_1, x_2, \ldots, x_m) \in D. \end{aligned}$$

We conclude that $H\big(x_1, x_2, \ldots, x_m, \Phi(x_1, x_2, \ldots, x_m)\big) = \text{const} = C$, for all $(x_1, x_2, \ldots, x_m) \in D$. Because of (3.30), $C = 0$, so we have

$$H\big(x_1, x_2, \ldots, x_m, \Phi(x_1, x_2, \ldots, x_m)\big) = 0,\ \forall\, (x_1, x_2, \ldots, x_m) \in D.$$

All conclusions of the Theorem 3.3 are proved. □

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