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# A property of the convolution of two $\varphi$-convex functions

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ABSTRACT. The convolution of two $\varphi$-convex functions is estimated by means of an inequality of Hadamard type. The sharpness of this inequality is discussed. As consequence, few inequalities for the convolution of two classically convex functions are obtained, providing us with a possibility of comparing the convolution product of two functions with their ordinary product, in the case of a class of convex functions.

## 1. INTRODUCTION AND PRELIMINARY RESULTS

The analytical theory of convexity for functions origins in Jensen's results from [4] and was developed in order to emphasize the common properties of many geometrical objects having convex shapes (see, for example, [1] and [10]). The geometrical significance of the convexity has many consequences in the most surprising domains of mathematics, as one can see in [1], [5], [7] and [9].

Let us recall the main concept restricting only to real functions, defined on a closed interval, which is a convex set. A function $f : [a, b] \to \mathbb{R}$, with $[a, b] \subset \mathbb{R}$, is said to be convex if whenever $x \in [a, b]$, $y \in [a, b]$ and $t \in [0, 1]$, the following inequality holds

$$f\left(tx + (1-t)\, y\right) \leq tf(x) + (1-t)\, f(y)\,. \tag{1.1}$$

A function having this property is called in our paper a classically convex function. Largely applied, due to its geometrical significance, the inequality of Hadamard (see [3]) generated a wide range of directions of extension and a rich mathematical literature. It was first published in [3], before Jensen's formal foundation of the theory of convex functions, stating that a classically convex function $f$ satisfies

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\, dx \leq \frac{f(a)+f(b)}{2}\,. \tag{1.2}$$

Many discussions, generalizations and applications of these inequalities have been published during the last century in the mathematical literature (see, for example [2], [5], [6], [9]). The purpose of this paper is to derive inequalities of Hadamard type for the convolution of two $\varphi-$convex functions. Here, we use the concept of $\varphi$-convexity as is defined in [1], taking into account the critical remarks from [11] refering to the ideas from [12].

Let us consider a function $\varphi : [a, b] \to [a, b]$.

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**Definition 1.1.** A function $f : [a, b] \to \mathbb{R}$ is said to be $\varphi$ - convex on $[a, b]$ if for every two points $x \in [a, b]$, $y \in [a, b]$ and $t \in [0, 1]$ the following inequality holds:

$$f\left(t\varphi\left(x\right)+\left(1-t\right)\varphi\left(y\right)\right) \leq tf\left(\varphi\left(x\right)\right)+\left(1-t\right)f\left(\varphi\left(y\right)\right). \tag{1.3}$$

If function $\varphi$ is the identity then the previous definition gives the classical convexity. The next lemma, obtained in [2], will be useful in the sequel.

**Lemma 1.1.** *The following statements are equivalent:*
$(i)$ *Function $f$ is $\varphi$ - convex on $[a, b]$,*
$(ii)$ *For every $x, y \in [a, b]$, the function $g : [0, 1] \to \mathbb{R}$, defined by*

$$g\left(t\right) = f\left(t\varphi\left(x\right)+\left(1-t\right)\varphi\left(y\right)\right)$$

*is classically convex on $[0, 1]$.*

## 2. Inequalities for the convolution of two $\varphi$-convex functions

Let us consider the class of all functions $f : \mathbb{R} \to \mathbb{R}$, with $f(x) = 0$ for $x < 0$, which are convex on $[0, +\infty)$ and have their support in $[0, +\infty)$. The convolution product of two functions $f$ and $g$ with the above mentioned properties is defined by

$$\left(f*g\right)\left(x\right) = \int_0^x f\left(t\right)g\left(x-t\right)dt. \tag{2.4}$$

As one knows, the convolution product is commutative.

Let us suppose that the non-decreasing and continuous function $\varphi : \mathbb{R} \to \mathbb{R}$ has the support included in $[0, +\infty)$. The following theorem has the expected consequences referring to the convolution of two $\varphi$-convex functions.

**Theorem 2.1.** *Let us consider two functions $f, g : \mathbb{R} \to \mathbb{R}$, with $f(x) = g(x) = 0$ for $x < 0$, $\varphi$-convex on $[0, +\infty)$ and having their support in $[0, +\infty)$. The following inequality holds*

$$\int_{\varphi(0)}^{\varphi(x)} f\left(\tau\right)g\left(\varphi\left(x\right)-\tau\right)d\tau \leq \frac{\varphi\left(x\right)-\varphi\left(0\right)}{6}\left[M\left(0,x\right)+2N\left(0,x\right)\right], \tag{2.5}$$

*for every real number $x \geq 0$, where*

$$M\left(0,x\right) = f\left(\varphi\left(0\right)\right)g\left(\varphi\left(0\right)\right)+f\left(\varphi\left(x\right)\right)g\left(\varphi\left(x\right)\right), \tag{2.6}$$

$$N\left(0,x\right) = f\left(\varphi\left(0\right)\right)g\left(\varphi\left(x\right)\right)+f\left(\varphi\left(x\right)\right)g\left(\varphi\left(0\right)\right). \tag{2.7}$$

*If $\varphi\left(0\right) = 0$, then*

$$\begin{aligned} 2\varphi\left(x\right)f\left(\frac{\varphi\left(x\right)}{2}\right)g\left(\frac{\varphi\left(x\right)}{2}\right) \leq \\ \leq \left(f*g\right)\left(\varphi\left(x\right)\right)+\frac{\varphi\left(x\right)}{6}\left[2M\left(0,x\right)+N\left(0,x\right)\right], \end{aligned} \tag{2.8}$$

*for every real number $x \geq 0$.*

*Proof.* In order to prove (2.5) we use the property of $\varphi-$ convexity of the two given functions taking in (1.3) the points 0 and $x>0$ instead of $x$ and $y$ and $t\in[0,1]$ as it follows:

$$f\left(t\varphi\left(0\right)+\left(1-t\right)\varphi\left(x\right)\right)\leq tf\left(\varphi\left(0\right)\right)+\left(1-t\right)f\left(\varphi\left(x\right)\right), \tag{2.9}$$

$$g\left(\varphi\left(x\right)-\left[t\varphi\left(0\right)+\left(1-t\right)\varphi\left(x\right)\right]\right)\leq tg\left(\varphi\left(x\right)\right)+\left(1-t\right)g\left(\varphi\left(0\right)\right). \tag{2.10}$$

The multiplication side by side of inequalities (2.9) and (2.10) yields

$$\begin{aligned}&f\left(t\varphi\left(0\right)+\left(1-t\right)\varphi\left(x\right)\right)g\left(\varphi\left(x\right)-\left[t\varphi\left(0\right)+\left(1-t\right)\varphi\left(x\right)\right]\right)\leq\\&\leq t^{2}f\left(\varphi\left(0\right)\right)g\left(\varphi\left(x\right)\right)+t\left(1-t\right)M\left(0,x\right)+\left(1-t\right)^{2}f\left(\varphi\left(x\right)\right)g\left(\varphi\left(0\right)\right).\end{aligned}$$

Lemma 2 implies the integrability with respect to $t$ over $[0,1]$ of both sides of the previous inequality. Integrating it and using (2.6) and (2.7) one gets

$$\begin{aligned}&\int_{0}^{1}f\left(t\varphi\left(0\right)+\left(1-t\right)\varphi\left(x\right)\right)g\left(\varphi\left(x\right)-\left[t\varphi\left(0\right)+\left(1-t\right)\varphi\left(x\right)\right]\right)dt\leq\\&\leq\frac{1}{6}M\left(0,x\right)+\frac{1}{3}N\left(0,x\right).\end{aligned}$$

The substitution $\tau=t\varphi\left(0\right)+\left(1-t\right)\varphi\left(x\right)$ in the integral from the left side of the previous inequality leads to

$$\frac{1}{\varphi\left(0\right)-\varphi\left(x\right)}\int_{\varphi(x)}^{\varphi(0)}f\left(\tau\right)g\left(\varphi\left(x\right)-\tau\right)d\tau\leq\frac{1}{6}M\left(0,x\right)+\frac{1}{3}N\left(0,x\right).$$

By the multiplication of the last inequality by $\varphi\left(x\right)-\varphi\left(0\right)$ one gets

$$\int_{\varphi(0)}^{\varphi(x)}f\left(\tau\right)g\left(\varphi\left(x\right)-\tau\right)d\tau\leq\frac{\varphi\left(x\right)-\varphi\left(0\right)}{6}\left[M\left(0,x\right)+2N\left(0,x\right)\right].$$

In order to prove (2.8) we start with the following calculus, with $\varphi\left(0\right)=0$, for every $x>0$ and $t\in[0,1]$ using the convexity of the given functions:

$$\begin{aligned}&f\left(\frac{\varphi\left(x\right)}{2}\right)g\left(\frac{\varphi\left(x\right)}{2}\right)=\\&=f\left(\frac{t0+\left(1-t\right)\varphi\left(x\right)}{2}+\frac{\left(1-t\right)0+t\varphi\left(x\right)}{2}\right)\\&\times g\left(\frac{t0+\left(1-t\right)\varphi\left(x\right)}{2}+\frac{\left(1-t\right)0+t\varphi\left(x\right)}{2}\right)\leq\\&\leq\frac{1}{4}f\left(t0+\left(1-t\right)\varphi\left(x\right)\right)g\left(\left(1-t\right)0+t\varphi\left(x\right)\right)+\\&+\frac{1}{4}f\left(\left(1-t\right)0+t\varphi\left(x\right)\right)g\left(t0+\left(1-t\right)\varphi\left(x\right)\right)+\\&+\frac{1}{4}\left[tf\left(0\right)+\left(1-t\right)f\left(\varphi\left(x\right)\right)\right]\left[tg\left(0\right)+\left(1-t\right)g\left(\varphi\left(x\right)\right)\right]+\\&+\frac{1}{4}\left[\left(1-t\right)f\left(0\right)+tf\left(\varphi\left(x\right)\right)\right]\left[\left(1-t\right)g\left(0\right)+tg\left(\varphi\left(x\right)\right)\right].\end{aligned}$$

A direct calculus of the right side of this inequality results in

$$
\begin{aligned}
&f\left(\frac{\varphi(x)}{2}\right) g\left(\frac{\varphi(x)}{2}\right) \leq \\
&\leq \frac{1}{4} f(t0+(1-t)\varphi(x))\, g((1-t)\,0+t\varphi(x))+ \\
&+\frac{1}{4} f((1-t)\,0+t\varphi(x))\, g(t0+(1-t)\varphi(x))+ \\
&+\frac{1}{4}\left[t^2+(1-t)^2\right] M(0,x)+\frac{1}{2} t(1-t)\, N(0,x).
\end{aligned}
$$

Lemma 2 allows us to integrate with respect to $t$ both sides of this inequality over $[0,1]$, which leads to

$$
f\left(\frac{\varphi(x)}{2}\right) g\left(\frac{\varphi(x)}{2}\right) \leq \frac{1}{4} \int_0^1 f(t0+(1-t)\varphi(x))\, g((1-t)\,0+t\varphi(x))\, dt+
$$

$$
+\frac{1}{4} \int_0^1 f((1-t)\,0+t\varphi(x))\, g(t0+(1-t)\varphi(x))\, dt+\frac{1}{6} M(0,x)+\frac{1}{12} N(0,x).
$$

After the same substitution as in the case of inequality (2.5), one gets

$$
\begin{aligned}
f\left(\frac{\varphi(x)}{2}\right) g\left(\frac{\varphi(x)}{2}\right) \leq \frac{1}{4\varphi(x)}(f*g)(\varphi(x))+\frac{1}{4\varphi(x)}(g*f)(\varphi(x))+ \\
+\frac{1}{6} M(0,x)+\frac{1}{12} N(0,x).
\end{aligned}
$$

The commutativity of the convolution product, together with the multiplication by $2\varphi(x)$ of both sides of the last inequality give the required inequality (2.8). $\square$

**Remark 2.1.** If $\varphi(x)=x$ for $x \geq 0$, then the previous inequalities become

$$
(f*g)(x) \leq \frac{x}{6}[M(0,x)+2N(0,x)] \tag{2.11}
$$

$$
2xf\left(\frac{x}{2}\right) g\left(\frac{x}{2}\right) \leq (f*g)(x)+\frac{x}{6}[2M(0,x)+N(0,x)]. \tag{2.12}
$$

**Remark 2.2.** If $f(0)=g(0)=0$ and $\varphi(x)=x$ for $x \geq 0$, then the previous inequalities become

$$
(f*g)(x) \leq \frac{x}{6} f(x)\, g(x), \tag{2.13}
$$

$$
2xf\left(\frac{x}{2}\right) g\left(\frac{x}{2}\right) \leq (f*g)(x)+\frac{x}{3} f(x)\, g(x), \tag{2.14}
$$

which allows a comparison between the convolution product and the ordinary product of two functions.

**Remark 2.3.** Both inequality (2.11) and (2.12) are sharp in the case of the functions that are polynomials of first degree on $\mathbb{R}_+$. Indeed, if $\varphi(x) = x$ for $x \geq 0$ and the two functions are $f, g : \mathbb{R} \to \mathbb{R}$, with $f(x) = g(x) = 0$ for $x < 0$, $f(x) = mx + n$ for $x \geq 0$, $g(x) = px + q$ for $x \geq 0$, with $m, n, p, q \in \mathbb{R}$, then

$$M(0, x) = mpx^2 + (mq + np)x + 2nq,$$

$$N(0, x) = (mq + np)x + 2nq.$$

In this case, both sides of inequality (2.11) become equal to

$$\frac{mp}{6}x^3 + \frac{mq + np}{2}x^2 + nqx.$$

Also, inequality (2.12) is sharp in this case, because both of its sides equal to

$$\frac{mp}{2}x^3 + (mq + np)x^2 + 2nqx.$$

**Remark 2.4.** Obviously, the functions from Remark 6 have the property of making sharp inequalities (2.13) and (2.14). As consequence, inequalities (2.5) and (2.8) are sharp in this case as well.

It would be of interest to find inequalities comparing the convolution product of two functions and the ordinary product of functions for as most as possible extended class of functions. Inequalities for the ordinary product of two $\varphi-$convex functions are derived in [2], [6] and [9]. A discussion of their sharpness, depending on function $\varphi$, might identify a remarkable class of functions behaving, in the case of the $\varphi-$convexity, as the linearity in the classical case.

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