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Another General Fixed Point Principles

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ABSTRACT. In the terms of fixed point structures two general fixed point principle was given by I. A. Rus ([3], [4]). In this paper we establish another fixed point principles in terms of fixed point structures.

1. INTRODUCTION AND NOTATIONS

Let X be a nonempty set. We denote by $P(X)$ the set of all nonempty subset of X. If $Y \in P(X)$, than (Y) is the set of all mappings $f : Y \to Y$. Let be $I(f)$ the set of all inward subset of f, i.e.

$$I(f) = \{Z \in P(Y) : f(Z) \subset Z\}.$$

For $f : Y \to Y$ consider $F_f = \{x \in Y : f(x) = x\}$, the set of all fixed point for f, and $K_f = \{x \in Y : f(x) = 0\}$. We say that a set X has the fixed point property with respect to $M(X) \subset (X)$ if for every $f \in M(X)$ we have $F_f \neq \emptyset$.

Definition 1.1. *(Rus* [3], [4]*)* A triple $(X, S(X), M)$ is *a fixed point structure* on the set X if:

(S_1) $S(X) \subset P(X)$, $S(X) \neq \emptyset$;

(S_2) $M : P(X) \to \bigcup_{Y \in P(X)} (Y)$ is a multivalued mapping such that, if $Z \in P(Y) \cap I(Y)$ and $f \in M(Y)$ than

$$f|_Z \in M(Z);$$

(S_3) every $Y \in S(X)$ has the fixed point property with respect to $M(Y)$.

Example 1.1. [Mönch] Let X be a Banach space, $S = P_{cl,cv}(X)$ and $M(Y) = \{f : Y \to Y;\ f$ is continuous and for some $x_0 \in Y$ and for all $C \subset Y$ countable, the inclusion $C \subset \overline{cv}\{\{x_0\} \cup f(C)\}$ implies $\overline{C}$ compact $\}$. The triple $(X, S(X), M)$ is a fixed point structure.

Another examples of fixed point structures it can find in [3], [4], [8] and [2]. In fact, for any fixed point theorem it might to formulate a fixed point structure. The importance of this notion is the unitary point of view to fixed point theorem. In the following we remainder some definition (for details see [3] - [8]).

Definition 1.2. Let X be a set and $(X, S(X), M)$ a fixed point structure. Consider the set $Z \subset P(X)$ such that $S(X) \subset Z$ and the operators $\theta : Z \to [0, \infty)$, $\eta : P(X) \to P(X)$. We said that the pair (θ, η) is *compatible with fixed point structure* $(X, S(X), M)$ if:

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(C_1) $A \subset \eta(A)$ for every $A \in P(X)$;
(C_2) if $A \subset B$ then $\eta(A) \subset \eta(B)$ for any $A, B \in P(X)$;
(C_3) $\eta^2 = \eta$;
(C_4) $S(X) \subset \eta(Z) \subset Z$ and $\theta(\eta(A)) = \theta(A)$ for all $A \in Z$;
(C_5) $F_\eta \cap K_\theta \subset S(X)$.

Remark that a mapping $\eta : P(X) \to P(X)$ which verify $(C_1)-(C_3)$ is a closure operator. Example of pair compatible with a fixed point structure it can find in [3] and [4].

Definition 1.3. A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a *comparison function* if:
(φ_1) φ is monotone increasing, i.e. $t_1 \le t_2$ implies $\varphi(t_1) \le \varphi(t_2)$;
(φ_2) $(\varphi^n(t))_{n\in\mathrm{N}}$ converges to 0 for all $t \le 0$.

For a comparison function we have the next result:

Lemma 1.1. (Rus [7]) *If φ is a comparison function then*

$$\varphi(t) < t \text{ for all } t > 0.$$

In the sequel we make the next notations“:
- $P_b(X)$ is the set of all nonempty and bounded subsets of X;
- $P_{cp}(X)$ is the set of all nonempty and compact subsets of X;
- $P_{b,cl}(X)$ is the set of all nonempty, close and compact subsets of X;

Definition 1.4. (Rus [7]) Let (X, d) be a complete metric space. A mapping $\alpha : P_b(X) \to \mathbb{R}_+$ is called *a measure of noncompactness* on X if and only if
(α_1) $\alpha(A) = 0$ implies $\bar{A} \in P_{cp}(X)$,
(α_2) $\alpha(A) = \alpha(\bar{A})$ for all $A \in P_b(X)$,
(α_3) $A \subset B \Rightarrow \alpha(A) \le \alpha(B)$ for all $A, B \in P_b(X)$,
(α_4) if $A_n \in P_{b,cl}(X)$, $A_{n+1} \subset A_n$, $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} \alpha(A_n) = 0$, imply that $\bigcap\limits_{n\ge1} A_n \ne \emptyset$, and $\alpha\left(\bigcap\limits_{n\ge1} A_n\right) = 0$.

Definition 1.5. Let (X, d) be a complete metric space. A mapping $\alpha_P : P_b(X) \to \mathbb{R}_+$ is called *a Pasicki's measure of noncompactness* if satisfies the conditions $(\alpha_1) - (\alpha_3)$ and
(α_5) $\alpha_P(A \cup \{x\}) = \alpha_P(A)$ for all $A \in P_b(X)$, and $x \in X$.

Axioms from Definition 1.4 admit to consider measures of noncompactness which is not permited by Definition 1.5, see [7] for examples.

Definition 1.6. (Rus [7]) Let (X, d) be a metric space, α a measure of noncompactness and φ a comparison function. A mapping $f : X \to X$ is a (α, φ)-*contraction* if and only if

$$\alpha(f(A)) \le \varphi(\alpha(A)) \text{ for all } A \in I_b(f).$$

The next result is *The First General Fixed Point Principle.*

Theorem 1.1. (Rus [4]) *Let $(X, S(X), M)$ be a fixed point structure and (θ, η) a compatible pair with $(X, S(X), M)$. Let $Y \in \eta(Z)$ and $f \in M(Y)$. Assume that:*

(H_1) the map $\theta : Z \to [0,\infty)$ is such that for every sequence $\{A_n\}_{n\geq 1} \subset Z$ with $A_{n+1} \subset A_n$, $n \in \mathbb{N}$ and $\lim_{n\to\infty} \theta(A_n) = 0$, imply that

$$A_\infty = \bigcap_{n\geq 1} A_n \neq \emptyset,\ A_\infty \in Z \text{ and } \theta(A_\infty) = 0;$$

(H_2) f is a (θ,φ)-contraction.

Then

a) $F_f \neq \emptyset$;

b) if $F_f \in Z$ then $\theta(F_f) = 0$.

We remainder that a functional $\theta : Z \to [0,\infty)$ which verify (H_1) is called a functional with intersection property.

The next result is known as *The Second General Fixed Point Principle.*

Theorem 1.2. (Rus [4]) *Let $(X, S(X), M)$ be a fixed point structure and (θ,η) a compatible pair with $(X, S(X), M)$. Let $Y \in \eta(Z)$ and $f \in M(Y)$. Assume that*

(K_1) for any $A \in Z$ and $x \in X$ we have $A \cup \{x\} \in Z$ and $\theta(A \cup \{x\}) = \theta(A)$;

(K_2) inequality $\theta(f(A)) < \theta(A)$ holds for any $A \in I(f) \cap Z$ with $\theta(A) \neq 0$.

Then

a) $F_f \neq \emptyset$;

b) if $F_f \in Z$ then $\theta(F_f) = 0$.

In both this theorems, we can replace the hypothesis $Y \subset \eta(Z)$ and $f \in M(Y)$ with

$$Y \in F_\eta \text{ and } f \in M(Y) \text{ such that } f(Y) \in Z$$

2. Another general fixed point principle

For convenience, we introduce a new notion:

Definition 2.7. Let X be a nonempty set $Y, Z \in P(X)$, $\theta : Z \to [0,\infty)$ and $\eta : P(X) \to P(X)$ a map which verifies $(C_1) - (C_3)$. The map $f \in M(Y)$ is (θ,η)-*Mönch operator* if for some $x_0 \in Y$ and for any $A \in P(Y) \cap Z$, A countable, the equality

$$A = \eta(\{x_0\} \cup f(A)) \text{ implies } \theta(A) = 0.$$

Example 2.2. Let (X, d) be a metric space, $Y, Z \in P(X)$. Consider the mappings $f \in M(Y)$, $\theta : Z \to [0,\infty)$ and $\eta : P(X) \to P(X)$ where the last one verifies $(C_1) - (C_3)$. Suppose that (K_1) and (K_2) are true, then f is an (θ,η)-Mönch operator. Indeed, for $A \in P(Y) \cap Z$ with $\theta(A) \neq 0$ we have

$$\theta(A) = \theta(\eta(\{x_0\} \cup f(A))) = \theta(\{x_0\} \cup f(A)) = \theta(f(A)) < \theta(A)$$

This is impossible, so $\theta(A) = 0$.

Example 2.3. Let (X, d) be a metric space, $Y, Z \in P(X)$. Consider the mappings $f \in M(Y)$, $\theta : Z \to [0,\infty)$ and $\eta : P(X) \to P(X)$ where the last one verifies $(C_1) - (C_3)$. Suppose that (K_1) and (H_2) are true, then f is an (θ,η)-Mönch operator. Indeed, for $A \in I(f) \cap Z$ with we have

$$\theta(A) = \theta(\eta(\{x_0\} \cup f(A))) = \theta(\{x_0\} \cup f(A)) = \theta(f(A)) \leq \varphi(\theta(A))$$

This is a contradiction with $\varphi(\theta(A)) < \theta(A)$. So, $\theta(A) = 0$.

In general, if $\theta = \alpha_P$ is a Pasicki's measure of noncompactness then a (α_P, φ)-contraction or α_P-condensing is a (θ, η)-Mönch operator.

Example 2.4. Let (X, d) be a metric space, $x_0 \in X$ and the functional $\delta : P(X) \to [0, \infty)$, given by

$$\delta(A) = \sup\{d(a, b); a, b \in A\}.$$

Obviously, $\delta(A) = 0$ if and only if A has a single one element, i.e. $A = \{x_0\}$. Hence, $f \in M(X)$ is (δ, η)-Mönch operator implies $f(x) = x_0$ for all $x \in X$. So, f is (δ, η)-Mönch operator if and only if f is constant.

The main theorem of this paper is the next result:

Theorem 2.3. *Let $(X, S(X), M)$ be a fixed point structure and (θ, η) a compatible pair with $(X, S(X), M)$. Let $Y \subset \eta(Z)$ and $f \in M(Y)$. Assume that f is an (θ, η)-Mönch operator. Then*

a) $I(f) \cap S(X) \neq \emptyset$;
b) $F_f \neq \emptyset$.

Proof. Let $x_0 \in Y$. We use the following lemma

Lemma 2.2. *Let X be a nonempty set and $\eta : P(X) \to P(X)$ a map which satisfies $(C_1) - (C_3)$. Let $Y \in F_\eta$, $A \in P(Y)$ and $f : Y \to Y$. Then, there is a set $A_0 \subset Y$ for which the next affirmation are true:*

(L_1) *$A \subset A_0$;*
(L_2) *$A_0 \in F_\eta$;*
(L_3) *$A_0 \in I(f)$;*
(L_4) *$\eta(f(A_0) \cup A) = A_0$,*

and deduce there exists $A_0 \subset Y$ such that $A_0 \in F_\eta \cap I(f)$ and $A_0 = \eta(\{x_0\} \cup f(A_0))$. Since f is an (θ, η)-Mönch operator, results $\theta(A_0) = 0$. Hence $A_0 \in F_\eta \cap K_\theta$. But the pair (θ, η) is compatible with fixed point structure, therefore $A_0 \in S(X)$. So $A_0 \in S(X) \cap I(f)$, i.e. $I(f) \cap S(X) \neq \emptyset$.

From $f|_{A_0} \in M(A_0)$ and $A_0 \in S(X)$ results $F_f \neq \emptyset$. □

As a consequence, we have

Theorem 2.4. *Let $(X, S(X), M)$ be a fixed point structure and (θ, η) a compatible pair with $(X, S(X), M)$. Let $Y \subset \eta(Z)$ and $f \in M(Y)$. Assume that (H_2) and (K_1) hold. Then*

a) $I(f) \cap S(X) \neq \emptyset$;
b) $F_f \neq \emptyset$;
c) if $F_f \in Z$ then $\theta(F_f) = 0$.

Proof. If the statements (H_2) and (K_1) hold, then f is (θ, η)-Mönch operator, so the conclusions result by Theorem 2.3. □

Example 2.4 shows that Theorem 2.3 do not generalize Theorem 1.1.

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