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Numerical experiment with the embedded Runge-Kuta formulae of the 6th order to the 5th order

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Abstract. In this paper we devote oneself to the numerical experiment with the embedded Runge-Kutta formulae of the 6th order to the 5th order. We deal with an influence of changing maximum allowable local errors and inserting either y_n or y_n to the embedded formulae on the accuracy of the approximate solution. We try to verify an advantage of using empirically deriving constant. The numerical solutions of two particular examples by using programming language Pascal are shown.

1. Introduction

Researching some phenomena and processes and trying to gain their mathematical descriptions it is sometimes not possible to find a direct dependent among variables that describe these processes. But we can determine a relation among researching variables and speeds of their change using another independent variables. Therefore we can obtain differential equations.

A large number of the processes especially in economy, medicine, biology, chemistry and also a lot of technical processes have an exponential character. That is a reason why we started to deal with possibilities of determination as accurate as possible the approximate solution of the Cauchy initial problem for the ordinary differential equation $y' = f(x, y)$, which has the exponential character.

Let us consider the ordinary differential equation

$$y' = f(x, y) \tag{1.1}$$

with the initial condition

$$y(x_0) = y_0. \tag{1.2}$$

We assume, that there exist just one solution $y(x)$ of the problem (1.1) , (1.2) in the interval $[a, b]$, which has an exponential character.

By inserting Runge-Kutta formulae of the 5-th order to the Runge-Kutta formulae of the 6-th order [1] we obtain following expresion of the approximate solution of the Cauchy initial problem (1.1), (1.2)

$$y_{n+1} = y_n + \frac{1}{144}\left(9K_n^{[1]} + 40K_n^{[3]} + 20K_n^{[4]} + 30K_n^{[5]} + 35K_n^{[6]} + 10K_n^{[7]}\right),$$

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and from the 6-th order the embedded formula of the approximate solution of the problem (1.1) , (1.2) can be written in the form

$$y_{n+1} = \\ = y_n + \frac{1}{288}\left(19K_n^{[1]} + 75K_n^{[3]} + 50K_n^{[4]} + 50K_n^{[5]} + 75K_n^{[6]} - 9K_n^{[7]} + 28K_n^{[8]}\right),$$

where

$$\begin{aligned}
K_n^{[1]} &= h_n\, f(x_n, y_n),\\
K_n^{[2]} &= h_n\, f\left(x_n + \frac{1}{10}h_n,\ y_n + \frac{1}{10}K_n^{[1]}\right),\\
K_n^{[3]} &= h_n\, f\left(x_n + \frac{1}{5}h_n,\ y_n + \frac{1}{5}K_n^{[2]}\right),\\
K_n^{[4]} &= h_n\, f\left(x_n + \frac{2}{5}h_n,\ y_n - \frac{1}{5}K_n^{[1]} + \frac{2}{5}K_n^{[2]} + \frac{1}{5}K_n^{[3]}\right),\\
K_n^{[5]} &= h_n\, f\left(x_n + \frac{3}{5}h_n,\ y_n + \frac{31}{30}K_n^{[1]} - \frac{64}{30}K_n^{[2]} + \frac{43}{30}K_n^{[3]} + \frac{8}{30}K_n^{[4]}\right),\\
K_n^{[6]} &= h_n\, f\left(x_n + \frac{4}{5}h_n,\ y_n + \frac{2}{35}K_n^{[1]} + \frac{20}{35}K_n^{[2]} + \frac{3}{35}K_n^{[3]} - \frac{34}{35}K_n^{[4]} + \frac{37}{35}K_n^{[5]}\right),\\
K_n^{[7]} &= h_n\, f\left(x_n + h_n,\ y_n - \frac{20}{10}K_n^{[1]} + \frac{28}{10}K_n^{[2]} - \frac{18}{10}K_n^{[3]} + \frac{38}{10}K_n^{[4]} - \frac{25}{10}K_n^{[5]} + \right.\\
&\quad \left. + \frac{7}{10}K_n^{[6]}\right),\\
K_n^{[8]} &= h_n\, f\left(x_n + h_n,\ y_n - \frac{9440}{5880}K_n^{[1]} + \frac{11342}{5880}K_n^{[2]} - \frac{9302}{5880}K_n^{[3]} + \frac{25982}{5880}K_n^{[4]} - \right.\\
&\quad \left. - \frac{17175}{5880}K_n^{[5]} + \frac{4473}{5880}K_n^{[6]}\right),
\end{aligned}$$

h_n is step size (for $n \geq 0$), h_0 is initial step size.

We denote $E_{n+1} = |y_{n+1} - y_{n+1}|$, determining size of the interval containing the solution at the point x_{n+1}.

There are a lot of estimates for control step size during computation in literature. We used formula in the form [2]:

$$h_{n+1} = 0,9\ h_n \left(\frac{\delta}{|\,E_{n+1}|}\right)^{\frac{1}{6}}$$

where δ is maximum allowable local error.

We applied the numerical experiment with the embedded formulae to solving these particular Cauchy initial problems:

1. $y' = -30\,y,\ y\,(0) = \frac{1}{3}$. This initial problem has the exact solution $\tilde{y}\,(x) = \frac{1}{3}\ e^{-30x}$.

We chose initial step size $h_0 = 0,05$ and determined solution in the interval $\langle\, 0;\ 0,2\,\rangle$.

2. $y' = x + y,\quad y\,(0) = 1$. The exact solution is function $\tilde{y}\,(x) = 2\,e^x - x - 1$, the initial step size $h_0 = 0,05$, solution was determined in the interval $[0; 1]$.

In the example 1 we followed the influence of inserting y_n and y_n to the embedded formulae on the accuracy of finding approximate solution. In both examples we dealt with advantages of choice empirical constant in the form $0,9,\ 0,8,\ 0,5$ and we gradually chose these maximum allowable local errors δ:$10^{-3},\ 10^{-4},\ 10^{-5},\ 10^{-6}$.

2. Results

2.1. **Influence of inserting y_n and y_n to the embedded formulae.** (used only example 1):

a) If we *insert* y_n to the embedded formulae, the approximate solution in the form y_n gets more accurate in the following steps of computation. The exact solution at the points $x_1,\ x_2,\ x_3,\ x_4$ doesn't lie in the interval $[y_n, y_n]$ (it's placed under this interval), but then it belons to this interval (Table No 1a).

Table No 1a: **Influence of inserting y_n and y_n to embedded formulae**
Ex.1: $y' = -30y$, $y(0) = 1/3$ in the interval $\langle 0; 0,2\rangle$, inserting y_n, empirical constant 0,9, $\delta = 10^{-3}$

n	x_n	$\tilde{y}_n - y_n$	$\tilde{y}_n - y_n$	E_n	h_n
0	0				0,05
1	0,05	-0,00037997	-0,00148945	0,00110948	0,04422751
2	0,09422751	-0,00007434	-0,00131089	0,00123655	0,03842081
3	0,13264832	-0,00001060	-0,00126278	0,00125218	0,03336650
4	0,16595497	-0,00000066	-0,00125512	0,00125446	0,02886451
5	0,19481948	0,00000050	-0,00125435	0,00125484	0,02501354
6	0,22025448	0,00000043	-0,00125449	0,00125492	0,02167613

b) If we *insert* y_n to the embedded formulae, the approximate solution in the form y_n gets more accurate. The exact solution lies in the interval $[y_n, y_n]$ (Table No 1b).

Table No 1b: **Influence of inserting y_n and y_n to the embedded formulae**
Ex. 1: $y' = -30y$, $y(0) = 1/3$ in the interval $\langle 0; 0,2\rangle$, inserting y_n, empirical constant 0,9, $\delta = 10^{-3}$

n	x_n	$\tilde{y}_n - y_n$	$\tilde{y}_n - y_n$	E_n	h_n
0	0				0,05
1	0,05	-0,00037997	-0,00148945	0,00110948	0,04422751
2	0,09422751	0,00074117	-0,00049727	0,00123843	0,03841105
3	0,13263856	0,00109450	-0,00015988	0,00125438	0,03328846
4	0,16592703	0,00119880	-0,00005790	0,00125670	0,02884015
5	0,19476717	0,00123312	-0,00002398	0,00125710	0,02498495
6	0,21976717	0,00124596	-0,00001121	0,00125717	0,02164488

We can say that number and size of steps are independent on inserting values y_n or y_n. Also the interval of the solution E_n stays the same in every step (the differences are visible at the 4-th decimal place, if the desired degree of accuracy is 10^{-3}).

Above mentioned results also hold if we desire higher degree of accuracy 10^{-6} (the differences among the values of E_n are visible at the 7-th decimal place).

2.2. **Influence of the value of empirical constant to control the next step size.**

a) Let consider example 1, interval $[0; 0, 2]$, where *values of the approximate solution are changing very slowly.*

Computing by larger steps is more advantageous (too small step costs negligible change, so that moving practically does't exist and that is disadvantageous in practice). Therefore the most advantageous is empirical constant $0, 9$. If we desired lower degree of accuracy ($\delta = 10^{-3}$), also empirical constant $0, 8$ could be acceptable. It needs more steps, but it gives higher accuracy of finding approximate solution (Table No 2a).

Table No 2a: **Influence of the value of empirical constant**
Ex.1: $y' = -30y$, $y(0) = 1/3$ in the interval $\langle 0; 0, 2\rangle$, $h_0 = 0, 05$, inserting y_n, $\delta = 10^{-3}$

	0,9	0,8	0,5
$\tilde{y}_1 - y_1$ h_1	-0,00148945 0,04422751	-0,00148945 0,03931335	-0,00148945 0,02457084
$\tilde{y}_2 - y_2$ h_2	-0,00049727 0,03841105	-0,00047663 0,03060650	-0,00070393 0,01206477
$\tilde{y}_3 - y_3$ h_3	-0,00015988 0,03328846	-0,00018613 0,02380986	-0,00048993 0,00592399
$\tilde{y}_4 - y_4$ h_4	-0,0000579 0,02884015	-0,00009015 0,01852106	-0,00041015 0,00290877
$\tilde{y}_5 - y_5$ h_5	-0,00002398 0,02498495	-0,00005153 0,01440690	-0,00037588 0,00142825
$\tilde{y}_6 - y_6$ h_6		-0,00003341 0,01120661	-0,00036011 0,00070129
$\tilde{y}_7 - y_7$ h_7		-0,00002386 0,00871722	-0,00035261 0,00034435
$\tilde{y}_8 - y_8$ h_8		-0,00001837 0,00678081	-0,00034899 0,00016908
$\tilde{y}_9 - y_9$ h_9			-0,00034722 0,00008302
procedure was stopped at the point $x_{15} = 0, 09827217$			$h_{15} = 0, 00000116$

If desired degree of accuracy is higher ($\delta = 10^{-6}$), only empirical constant $0, 9$ is suitable for meaningful computing.

b) Let consider example 2, where *values of the approximate solution are changing slowly*:

The number of steps for constant $0, 9$ and $0, 8$ is almost the same, but using constant $0, 8$ we achieve higher accuracy of the approximate solution. Constant $0, 5$ makes steps two times smaller. These results we achieve in each degree of accuracy. Bigger differences are visible at higher accuracy 10^{-6} (Table No 2b).

Table No 2b: **Influence of the value of empirical constant**
Ex. 2: $y' = x + y$, $y(0) = 1$ in the interval $[0; 1]$, $h_0 = 0,05$, inserting y_n, $\delta = 10^{-6}$

	0,9	0,8	0,5
$\tilde{y}_1 - y_1$ h_1	0 0,20015407	0 0,17791473	0 0,11119671
$\tilde{y}_2 - y_2$ h_2	-0,00000118 0,25517555	-0,00000065 0,22168509	0,00000006 0,12661081
$\tilde{y}_3 - y_3$ h_3	-0,00000657 0,24948429	-0,00000319 0,21537872	0,00000019 0,11955067
$\tilde{y}_4 - y_4$ h_4	-0,00001422 0,21873058	-0,00000653 0,18899217	0,00000033 0,10461958
$\tilde{y}_5 - y_5$ h_5	-0,00002147 0,18291429	-0,00000952 0,15870679	0,00000043 0,08863646
$\tilde{y}_6 - y_6$ h_6		-0,00001196 0,13078187	0,00000049 0,07406207
$\tilde{y}_1 - y_1$ h_1			0,00000055 0,07406207
procedure was stopped at the point $x_{15} = 0,95319308$			$h_{15} = 0,01352552$

3. Conclusion

In this numerical experiment with the embedded Runge-Kutta formulae of the 6-th order to the 5-th order we followed influence of changing initial values on the accuracy of the approximate solution of the Cauchy problem for the ordinary differential equation. We devoted oneself to solutions with exponential character. Inserting values of y_n or y_n has no influence to number and size of computing steps. Also the interval E_n we can consider the same in every step. Empiric constant 0, 5 makes step too small and it cased negligible mooving what is disadvantageous in practice. Empiric constant 0,8 needs more steps than the empiric constant 0, 9 (it cased smaller step), but it is more advantageous than empiric constant 0, 9 if the desired degree of accuracy is lower ($\delta = 10^{-3}$). Empirically denoted constant 0, 9 is in general the most advantageous taking into account the number of steps and protecting desired accuracy of the solution.

References

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