CARPATHIAN J. MATH.
20 (2004), No. 1, 73 - 80

Very well-covered graphs with log-concave independence polynomials

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ABSTRACT. If s_k equals the number of stable sets of cardinality k in the graph G, then $I(G;x) = \sum_{k=0}^{\alpha(G)} s_k x^k$ is the *independence polynomial* of G (Gutman and Harary, 1983). Alavi, Malde, Schwenk and Erdös (1987) conjectured that $I(G;x)$ is unimodal whenever G is a forest, while Brown, Dilcher and Nowakowski (2000) conjectured that $I(G;x)$ is unimodal for any well-covered graph G. Michael and Traves (2002) showed that the assertion is false for well-covered graphs with $\alpha(G) \geq 4$, while for very well-covered graphs the conjecture is still open.

In this paper we give support to both conjectures by demonstrating that if $\alpha(G) \leq 3$, or $G \in \{K_{1,n}, P_n : n \geq 1\}$, then $I(G^*;x)$ is log-concave, and, hence, unimodal (where G^* is the very well-covered graph obtained from G by appending a single pendant edge to each vertex).

1. INTRODUCTION

Throughout this paper $G = (V, E)$ is a finite, undirected, loopless and without multiple edges graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The set $N(v) = \{u : u \in V, uv \in E\}$ is the *neighborhood* of $v \in V$, and $N[v] = N(v) \cup \{v\}$. As usual, a *tree* is an acyclic connected graph, while a *spider* is a tree having at most one vertex of degree ≥ 3. $K_n, P_n, K_{n_1,n_2,...,n_p}$ denote, respectively, the complete graph on $n \geq 1$ vertices, the chordless path on $n \geq 1$ vertices, and the complete p-partite graph on $n_1 + n_2 + ... + n_p$ vertices, $n_1, n_2, ..., n_p \geq 1$. A graph is called *claw-free* if it has no induced subgraph isomorphic to $K_{1,3}$. The *disjoint union* of the graphs G_1, G_2 is the graph $G = G_1 \sqcup G_2$ having $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$. If G_1, G_2 are disjoint graphs, then their *Zykov sum*, ([20]), is the graph $G_1 \uplus G_2$ with $V(G_1 \uplus G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \uplus G_2) = E(G_1) \cup E(G_2) \cup \{v_1v_2 : v_1 \in V(G_1), v_2 \in V(G_2)\}$. In particular, $\sqcup nG$ and $\uplus nG$ denote the disjoint union and Zykov sum, respectively, of $n > 1$ copies of the graph G.

A *stable* set in G is a set of pairwise non-adjacent vertices. The *stability number* $\alpha(G)$ of G is the maximum size of a stable set in G. A graph G is called *well-covered* if all its maximal stable sets are of the same cardinality, [18]. If, in addition, G has no isolated vertices and its order equals $2\alpha(G)$, then G is *very well-covered*, [4]. By G^* we mean the graph obtained from G by appending a single pendant edge to each vertex of G. Let us remark that G^* is well-covered (see, for instance, [9]), and $\alpha(G^*) = n$. In fact, G^* is very well-covered.

Received: 26.09.2004; In revised form: 12.01.2005
2000 *Mathematics Subject Classification.* Primary 05C69, 05E99; Secondary 11B83, 05A20.
Key words and phrases. *well-covered graph, stable set, unimodality, independence polynomial, log-concavity, claw-free graph, tree.*

Let s_k be the number of stable sets in G of cardinality $k \in \{0, 1, ..., \alpha(G)\}$. The polynomial $I(G; x) = \sum_{k=0}^{\alpha(G)} s_k x^k = 1 + s_1 x + s_2 x^2 + ... + s_\alpha x^\alpha, \alpha = \alpha(G)$ is called the *independence polynomial* of G, [6]). In [6] was also proved the following equalities.

Proposition 1.1. *If $v \in V(G)$, then $I(G; x) = I(G - v; x) + xI(G - N[v]; x)$, and*

$$I(G_1 \sqcup G_2; x) = I(G_1; x) \cdot I(G_2; x), \quad I(G_1 \uplus G_2; x) = I(G_1; x) + I(G_2; x) - 1.$$

A finite sequence of real numbers $(a_0, a_1, a_2, ..., a_n)$ is said to be *unimodal* if there is some k, called the *mode* of the sequence, such that $a_0 \leq ... \leq a_{k-1} \leq a_k \geq a_{k+1} \geq ... \geq a_n$, and *log-concave* if $a_i^2 \geq a_{i-1} \cdot a_{i+1}$ for $1 \leq i \leq n-1$. It is known that any log-concave sequence of positive numbers is also unimodal. A polynomial is called *unimodal (log-concave)* if the sequence of its coefficients is unimodal (log-concave, respectively). For instance, $I(K_n \uplus (\sqcup 3K_7); x) = 1 + (n + 21)x + 147x^2 + 343x^3, n \geq 1$, is *(a)* log-concave, if $147^2 - (n+21) \cdot 343 \geq 0$, i.e., for $1 \leq n \leq 42$ (e.g., $I(K_{42} \uplus (\sqcup 3K_7); x) = 1 + 63x + 147x^2 + 343x^3$), *(b)* unimodal, but non-log-concave, whenever $147^2 - (n + 21) \cdot 343 < 0$ and $n \leq 126$, that is, $43 \leq n \leq 126$ (for instance, $I(K_{43} \uplus (\sqcup 3K_7); x) = 1 + 64x + 147x^2 + 343x^3$), *(c)* non-unimodal for $n \geq 127$ (e.g., $I(K_{127} \uplus (\sqcup 3K_7); x) = 1 + 148x + 147x^2 + 343x^3$). The graph $H = (\sqcup 3K_{10}) \uplus K_{\underbrace{3,3,...,3}_{120}}$ is connected and well-covered, but not very well-covered, and its independence polynomial is unimodal, but not log-concave: $I(H; x) = 1 + 390x + 660x^2 + \mathbf{1120}x^3$. The product of two polynomials, one log-concave and the other unimodal, is not always log-concave, for instance, if $G = K_{40} \uplus (\sqcup 3K_7), H = K_{110} \uplus (\sqcup 3K_7)$, then

$$\begin{aligned} I(G; x) \cdot I(H; x) &= \left(1 + 61x + 147x^2 + 343x^3\right)\left(1 + 131x + 147x^2 + 343x^3\right) \\ &= 1 + 192x + 8285x^2 + 28910x^3 + 87465x^4 + 100842x^5 + 117649x^6. \end{aligned}$$

However, the following result, due to Keilson and Gerber, states that:

Theorem 1.1. [8] *If $P(x)$ is log-concave and $Q(x)$ is unimodal, then $P(x) \cdot Q(x)$ is unimodal, while the product of two log-concave polynomials is log-concave.*

Alavi *et al.* [1] showed that for any permutation σ of $\{1, 2, ..., \alpha\}$ there is a graph G with $\alpha(G) = \alpha$ such that $s_{\sigma(1)} < s_{\sigma(2)} < ... < s_{\sigma(\alpha)}$. Nevertheless, in [1] it is stated the following (still open) conjecture: $I(F; x)$ of any forest F is unimodal.

In [2] it was conjectured that $I(G; x)$ is unimodal for each well-covered graph G. Michael and Traves [17] proved that this assertion is true for $\alpha(G) \leq 3$, but it is false for $4 \leq \alpha(G) \leq 7$. In [15] we showed that for any $\alpha \geq 8$, there exists a connected well-covered graph G with $\alpha(G) = \alpha$, whose $I(G; x)$ is not unimodal. However, the conjecture of Brown *et al.* is still open for very well-covered graphs. In [14] an infinite family of very well-covered graphs with unimodal independence polynomials is described. We also showed that $I(G^*; x)$ is unimodal for any G^* whose skeleton G has $\alpha(G) \leq 4$ (see [14]).

Michael and Traves [17] formulated (and verified for well-covered graphs with stability numbers ≤ 7) the following so-called "*roller-coaster*" conjecture: for any permutation π of the set $\{\lceil \alpha/2 \rceil, \lceil \alpha/2 \rceil + 1, ..., \alpha\}$, there exists a well-covered graph G, with $\alpha(G) = \alpha$, whose sequence $(s_0, s_1, ..., s_\alpha)$ satisfies the inequalities

$s_{\pi(\lceil\alpha/2\rceil)} < s_{\pi(\lceil\alpha/2\rceil+1)} < ... < s_{\pi(\alpha)}$. Recently, Matchett [16] showed that this conjecture is true for well-covered graphs with stability numbers ≤ 11.

Recall also the following statement, due to Hamidoune.

Theorem 1.2. [7] *The independence polynomial of a claw-free graph is log-concave.*

As a consequence, we deduce that for any $\alpha \geq 1$, there exists a tree T, with $\alpha(T) = \alpha$ and whose $I(T;x)$ is log-concave, e.g., the chordless path $P_{2\alpha}$.

In this paper we show that the independence polynomial of G^* is log-concave, whenever: $\alpha(G) \leq 3$, or G^* is a well-covered spider (i.e., $G = K_{1,n}, n \geq 1$), or G^* is a centipede (that is, $G = P_n, n \geq 1$).

2. Results

Lemma 2.1. *If G is a graph of order $n \geq 1$ and $\alpha(G) = \alpha$, then $\alpha \cdot s_\alpha \leq n \cdot s_{\alpha-1}$.*

Proof. Let $H = (\mathcal{A}, \mathcal{B}, \mathcal{W})$ be the bipartite graph defined as follows: $X \in \mathcal{A} \Leftrightarrow X$ is a stable set in G of size $\alpha - 1$, then $Y \in \mathcal{B} \Leftrightarrow Y$ is a stable set in G of size $\alpha(G)$, and $XY \in \mathcal{W} \Leftrightarrow X \subset Y$ in G. Since any $Y \in \mathcal{B}$ has exactly $\alpha(G)$ subsets of size $\alpha - 1$, it follows that $|\mathcal{W}| = \alpha \cdot s_\alpha$. On the other hand, if $X \in \mathcal{A}$, then $|\{X \cup \{y\} : X \cup \{y\} \in \mathcal{B}\}| \leq n - |X| = n - \alpha + 1$. Hence, any $X \in \mathcal{A}$ has at most $n - \alpha + 1$ neighbors. Consequently, $|\mathcal{W}| = \alpha \cdot s_\alpha \leq (n - \alpha + 1) \cdot s_{\alpha-1}$, and this leads to $\alpha \cdot s_\alpha \leq n \cdot s_{\alpha-1}$. □

In [13] it was established the following result:

Theorem 2.3. [13] *If G is a graph of order $n \geq 1$ and $I(G;x) = \sum_{k=0}^{\alpha(G)} s_k x^k$, then*

$$I(G^*;x) = \sum_{k=0}^{\alpha(G^*)} t_k x^k, \quad t_k = \sum_{j=0}^{k} s_j \cdot \binom{n-j}{n-k}, \quad 0 \leq k \leq \alpha(G^*) = n.$$

In [14] it was shown that $I(G^*;x)$ is unimodal for any graph G with $\alpha(G) \leq 4$. Now we partially strengthen this assertion to the following result.

Theorem 2.4. *If G is a graph with $\alpha(G) \leq 3$, then $I(G^*;x)$ is log-concave.*

Proof. Suppose that $\alpha(G) = 3$. Then $n = |V(G)| \geq 3$ and $I(G;x) = 1 + nx + s_2x^2 + s_3x^3$. According to Theorem 2.3, for $2 \leq k \leq n-1$, we obtain: $t_k = \binom{n}{k} + n\binom{n-1}{k-1} + s_2\binom{n-2}{k-2} + s_3\binom{n-3}{k-3}$. Therefore,

$$\begin{aligned} t_k^2 - t_{k-1}t_{k+1} &= A_0 + n^2A_1 + s_2^2A_2 + s_3^2A_3 + \\ &\quad nA_{01} + s_2A_{02} + s_3A_{03} + ns_2A_{12} + ns_3A_{13} + s_2s_3A_{23}, \end{aligned}$$

and all $A_i \geq 0, 0 \leq i \leq 3$, where

$$\begin{aligned} A_0 &= \binom{n}{k}^2 - \binom{n}{k-1}\binom{n}{k+1}, \quad A_1 = \binom{n-1}{k-1}^2 - \binom{n-1}{k-2}\binom{n-1}{k}, \\ A_2 &= \binom{n-2}{k-2}^2 - \binom{n-2}{k-3}\binom{n-2}{k-1}, \quad A_3 = \binom{n-3}{k-3}^2 - \binom{n-3}{k-4}\binom{n-3}{k-2}, \end{aligned}$$

because the sequence $\left\{\binom{n}{k}\right\}$ is log-concave. Based on notation $b=\binom{n}{k}^2$, we get

$$\begin{aligned}
A_{01} &= \frac{2k\,(n+1)\,b}{n\,(n-k+1)\,(k+1)}, \quad A_{02}=\frac{2kb\{(k-2)n+2k-1\}}{(k+1)\,(n-k+1)\,(n-1)\,n},\\
A_{03} &= \frac{2kb(k-1)\{(k-5)n+4k-2\}}{(k+1)\,(n-k+1)\,(n-2)\,(n-1)\,n}, \quad A_{12}=\frac{2kb(k-1)}{n(n-1)\,(n-k+1)},\\
A_{13} &= \frac{2kb\,(k-1)\,\{(k-3)n+k\}}{(n-2)\,(n-1)\,n^2(n-k+1)}, \quad A_{23}=\frac{2k^2b\,(k-1)\,(k-2)}{(n-2)\,(n-1)\,n^2\,(n-k+1)},
\end{aligned}$$

and all $A_{ij}\geq 0$ for $k\geq 5$. Hence, we must check that $t_k^2-t_{k-1}t_{k+1}\geq 0$ for $k\in\{1,2,3,4\}$. By Theorem 2.3, we obtain:

$$t_0=1,t_1=2n,t_2=3n(n-1)/2+s_2,t_3=\frac{2n(n-1)(n-2)}{3}+(n-2)s_2+s_3,$$

$$t_4=\frac{5}{24}\,(n-3)\,n\,(n-1)\,(n-2)+\frac{1}{2}s_2\,(n-2)\,(n-3)+s_3n-3s_3,$$

$$t_5=(n-4)\,(n-3)\left[\frac{1}{20}n\,(n-1)\,(n-2)+\frac{1}{6}s_2\,(n-2)+\frac{1}{2}s_3\right].$$

Consequently, it follows $t_1^2-t_0t_1=\left(n^2+2\left(2n^2-s_2\right)\right)/2>0$. We also deduce

$$t_2^2-t_1t_3=\frac{1}{12}\,(11n+5)\,(n-1)n^2+s_2^2+ns_2+n\,(ns_2-2s_3)\geq 0,$$

since $3s_3\leq ns_2$ is true according to Lemma 2.1.

Now, simple calculations lead us to

$$\begin{aligned}
&144\left(t_3^2-t_2t_4\right)=(19n+7)\,n^2\,(n-1)^2\,(n-2)+(54n+30)\,n\,(n-1)\,(n-2)\,s_2\\
&-24\,(n-11)\,n\,(n-1)\,s_3+72n\,(n-3)\,s_2^2+144(s_3^2+(n-1)s_2s_3+s_2^2).
\end{aligned}$$

Let us notice that $n\,(n-1)\,((54n+30)\,(n-2)\,s_2-24\,(n-11)\,s_3)\geq 0$, because Lemma 2.1 implies the inequality $54ns_2\geq 24s_3$. Hence, we infer that $t_3^2-t_2t_4\geq 0$, whenever $n\geq 3$.

Further, we have

$$\begin{aligned}
&2880\left(t_4^2-t_3t_5\right)=\left(29n^8-252n^7-108n^2+818n^6+12n^3-1200n^5+701n^4\right)+\\
&+\left(672n+2680n^4-2520n^3+64n^2+136n^6-1032n^5\right)s_2+\\
&+\left(-3840n^3+8400n^2-4896n+96n^5+240n^4\right)s_3+\\
&+\left(240n^4-1920n^3+5520n^2-6720n+2880\right)s_2^2+\\
&+\left(10\,560n-5760n^2+960n^3-5760\right)s_3s_2+\left(8640+1440n^2-7200n\right)s_3^2\\
&=(29n+9)\,n^2\,(n-1)^2\,(n-2)^2\,(n-3)+\\
&+(136n+56)\,n\,(n-1)\,(n-2)^2\,(n-3)\,s_2+\\
&+(96n+816)\,n\,(n-1)\,(n-2)\,(n-3)\,s_3+240\,(n-1)\,(n-2)^2\,(n-3)\,s_2^2+\\
&+960\,(n-1)\,(n-2)\,(n-3)\,s_2s_3+1440\,(n-2)\,(n-3)\,s_3^2\geq 0.
\end{aligned}$$

Consequently, $t_k^2-t_{k-1}t_{k+1}\geq 0$, for $1\leq k\leq n-1$, i.e., $I(G^*;x)$ is log-concave.

The log-concavity for the cases $\alpha(G)\in\{1,2\}$ can be validated in a similar way, by observing that either $s_2=s_3=0$ or only $s_3=0$. □

Since $\alpha(K_{1,n})=n,\alpha(P_n)=\lceil n/2\rceil$, Theorem 2.4 is not useful in proving that $I(K_{1,n}^*;x)$, $I(P_n^*;x)$ are log-concave, as soon as n is sufficiently large. In [11],

[12] we proved that $I(K_{1,n}^{*};x)$, $I(W_n;x)$ are unimodal. Here we are strengthening these results.

The well-covered spider $S_n, n \geq 2$, has n vertices of degree 2, one vertex of degree $n+1$, and $n+1$ vertices of degree 1 (see Figure 1). In fact, it is easy to see that $S_n = K_{1,n}^{*}, n \geq 2$.

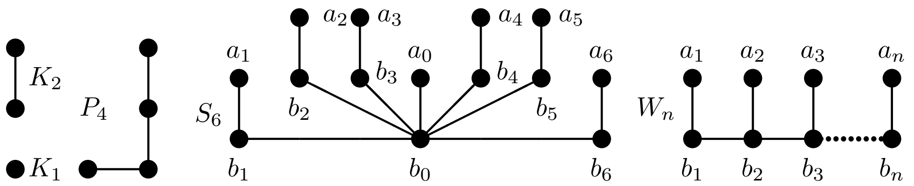

FIGURE 1. Well-covered spiders: K_1, K_2, P_4, S_6, and the centipede W_n.

Proposition 2.2. [12] *The independence polynomial of any well-covered spider is unimodal, moreover,* $I(S_n;x) = (1+x) \cdot \sum\limits_{k=0}^{n} \left[\binom{n}{k} \cdot 2^k + \binom{n-1}{k-1} \right] \cdot x^k, n \geq 2$, *and its mode is unique and equals* $1 + (n-1)$ ***mod*** $3 + 2\left(\lceil n/3 \rceil - 1\right)$.

In [2] it was shown that $I(G;x)$ of any graph G with $\alpha(G) = 2$ has real roots, and, hence, it is log-concave, according to Newton's theorem (stating that if a polynomial with positive coefficients has only real roots, then its coefficients form a log-concave sequence). However, Newton's theorem is not useful in solving the conjecture of Alavi *et al.*, even for the particular case of very well-covered trees, since, for instance, $I(S_3;x) = 1 + 8x + 21x^2 + 23x^3 + 9x^4$ has non-real roots.

Theorem 2.5. *The independence polynomial of any well-covered spider is log-concave.*

Proof. Since $I(G;x)$ is log-concave for any graph G with $\alpha(G) \leq 2$, we consider only well-covered spiders S_n with $n \geq 2$. According to Proposition 2.2,

$$I(S_n;x) = (1+x) \cdot \sum_{k=0}^{n} \left[\binom{n}{k} \cdot 2^k + \binom{n-1}{k-1} \right] \cdot x^k = (1+x) \cdot P(x).$$

It is sufficient to prove that $P(x)$ is log-concave, because, further, Theorem 1.1 implies that $I(S_n;x)$ is log-concave, as well. Let us denote $c_k = \binom{n}{k} \cdot 2^k + \binom{n-1}{k-1}, 0 \leq k \leq n$.

Firstly, we notice that $c_1^2 - c_0 \cdot c_2 = (2n+1)(n+2) > 0$. Further, for $2 \leq k \leq n-1$, we obtain that:

$$\begin{aligned} c_k^2 - c_{k-1} \cdot c_{k+1} &= \left[\binom{n-1}{k-1}^2 - \binom{n-1}{k-2}\binom{n-1}{k} \right] + \\ &+ \binom{n}{k}^2 \frac{n(2n+2)2^k - k^2(n+3) + k(k^2+7n+4)}{n(k+1)(n-k+1) \cdot 2^{1-k}}. \end{aligned}$$

Clearly, $\binom{n-1}{k-1}^2 - \binom{n-1}{k-2}\binom{n-1}{k} \geq 0$, since the sequence of binomial coefficients is log-concave, and $n(2n+2)2^k - k^2(n+3) \geq 0$, because $n \cdot 2^k \geq k^2$ holds for any $k \in \{2, ..., n-1\}$. Thus, $c_k^2 - c_{k-1} \cdot c_{k+1} \geq 0$, for any $k \in \{1, 2, ..., n-1\}$. □

The *edge-join* of two disjoint graphs G_1, G_2, is the graph $G_1 \ominus G_2$ obtained by adding an edge joining a vertex from G_1 to a vertex from G_2. If both vertices are of degree at least two, then $G_1 \ominus G_2$ is an *internal edge-join* of G_1, G_2. By $\triangle_n$ we mean the graph $\ominus nK_3 = (\ominus(n-1)K_3) \ominus K_3, n \geq 1$ (see Figure 2).

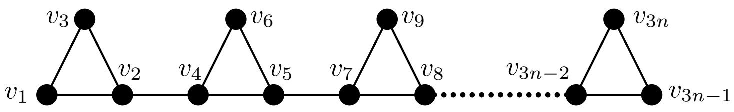

FIGURE 2. The graph $\triangle_n = (\ominus(n-1)K_3) \ominus K_3$.

In [5] it is shown that apart from K_1 and C_7, any connected well-covered graph of girth ≥ 6 equals G^* for some graph G; e.g., every well-covered tree equals T^* for some tree T (see also [19]). Thus, a tree $T \neq K_1$ could be only very well-covered.

Theorem 2.6. [10] *A tree T is well-covered if and only if T is a well-covered spider, or T is the internal edge-join of a number of well-covered spiders.*

A *centipede* is a well-covered tree defined by $W_n = P_n^*, n \geq 1$ (see Figure 1). For example, $W_1 = K_2, W_2 = P_4, W_3 = S_2$.

Theorem 2.7. *The independence polynomial of any centipede is log-concave.*

Proof. We show, by induction on $n \geq 1$, that

$$I(W_{2n};x) = (1+x)^n \cdot I(\triangle_n;x), \quad I(W_{2n+1};x) = (1+x)^n \cdot I(\triangle_n \ominus K_2;x),$$

(for another proof of these equalities, see [12]).

For $n = 1$, the assertion is true, because

$$\begin{aligned} I(W_2;x) &= 1 + 4x + 3x^2 = (1+x)(1+3x) = (1+x) \cdot I(\triangle_1;x), \\ I(W_3;x) &= 1 + 6x + 10x^2 + 5x^3 = (1+x) \cdot I(\triangle_1 \ominus K_2;x). \end{aligned}$$

Assume that the formulae are true for $k \leq 2n+1$. By Proposition 1.1, we get:

$$\begin{aligned} I(W_{2n+2};x) &= I(W_{2n+2} - b_{2n+1};x) + x \cdot I(W_{2n+2} - N[b_{2n+1}];x) \\ &= (1+x)(1+2x) \cdot I(W_{2n};x) + x(1+x)^2 \cdot I(W_{2n-1};x) \\ &= (1+x)^{n+1} \cdot \{I(K_2;x) \cdot I(\triangle_n;x) + x \cdot I(\triangle_{n-1} \ominus K_2;x))\}. \end{aligned}$$

On the other hand, if v is the vertex of degree 3 in the last triangle of $\triangle_{n+1}$(see Figure 3(a)), then $I(\triangle_{n+1};x) = I(K_2;x)I(\triangle_n;x) + xI(\triangle_{n-1} \ominus K_2;x))$, according to Proposition 1.1. In other words, $I(W_{2n+2};x) = (1+x)^{n+1} \cdot I(\triangle_{n+1};x)$.

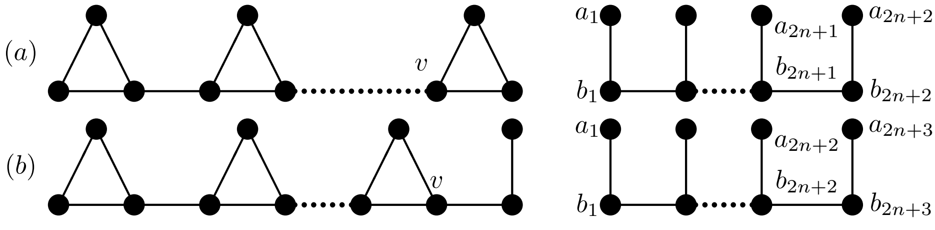

FIGURE 3. The graphs: (a) $\triangle_{n+1}$ and W_{2n+2}; (b) $\triangle_{n+1} \ominus K_2$ and W_{2n+3}.

Similarly, again by Proposition 1.1, we obtain:

$$\begin{aligned} I(W_{2n+3};x) &= I(W_{2n+3}-b_{2n+2};x)+x\cdot I(W_{2n+3}-N[b_{2n+2}];x) \\ &= (1+x)\,(1+2x)\cdot I(W_{2n+1};x)+x\,(1+x)^2\cdot I(W_{2n};x) \\ &= (1+x)^{n+1}\left\{I(K_2;x)\cdot I(\triangle_n\ominus K_2;x)+x\,(1+x)\cdot I(\triangle_n;x))\right\}. \end{aligned}$$

On the other hand, if v is the vertex of degree 3 belonging to the last triangle of $\triangle_{n+1}\ominus K_2$ (see Figure 3(b)) and adjacent to one of the vertices of K_2, we have

$$\begin{aligned} I(\triangle_{n+1}\ominus K_2;x) &= I(\triangle_{n+1}\ominus K_2-v;x)+xI(\triangle_{n+1}\ominus K_2-N[v];x) \\ &= I(K_2;x)\cdot I(\triangle_n\ominus K_2;x)+x\,(1+x)\cdot I(\triangle_n;x)). \end{aligned}$$

In other words,

$$I(W_{2n+3};x)=(1+x)^{n+1}\cdot I(\triangle_{n+1}\ominus K_2;x).$$

While Theorem 1.2 assures that $I(\triangle_n;x),I(\triangle_n\ominus K_2;x)$ are log-concave, finally Theorem 1.1 implies that $I(W_n;x)$ is log-concave, as claimed. □

Corollary 2.1. (i) *If the graph H has as connected components well-covered spiders/centipedes and/or graphs with stability number ≤ 2, and/or claw-free graphs, and/or graphs that may be represented as G^* whose G has $\alpha(G)\leq 3$, then its independence polynomial $I(H;x)$ is log-concave.*
(ii) *If $H_n\in\{S_n,W_n\}$, then the independence polynomial of $\uplus mH_n$ is log-concave, for any $m\geq 2,n\geq 1$.*

Proof. *(i)* Let $G_i,1\leq i\leq m$, be the connected components of G. According to Theorems 2.5, 2.7, 2.4 and 1.2, any $I(G_i;x)$ is log-concave. Further, Theorem 1.1 implies that $I(G;x)$ is also log-concave, as $I(G;x)=I(G_1;x)\cdot\ldots\cdot I(G_m;x)$.

(ii) Since $I(H_n;x)$ is log-concave, and $I(\uplus mH_n;x)=m\cdot I(H_n;x)-(m-1)$, it follows that $I(\uplus mH_n;x)$ is log-concave, as well. □

3. Conclusions

In this paper we showed that for any α, there is a very well-covered tree T with $\alpha(T)=\alpha$, whose independence polynomial $I(T;x)$ is log-concave. We conjecture that the independence polynomial of any (well-covered) forest is log-concave.

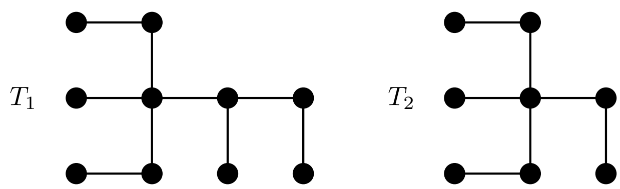

Figure 4. Two (very) well-covered trees.

In 1990, Hamidoune [7] conjectured that the independence polynomial of any claw-free graph has only real roots. Recently, Chudnovsky and Seymour [3] validated this conjecture. Consequently, $I(P_n;x)$ has all the roots real. Moreover, the roots of $I(W_n;x)$ are real (see the proof of Theorem 2.7).

For general (very well-covered) spiders/trees the structure of the roots of the independence polynomial is more complicated. For instance, the independence polynomial of the claw graph $I(K_{1,3};x) = 1 + 4x + 3x^2 + x^3$ has non-real roots. Figure 4 provides us with some more examples:

$$\begin{aligned} I(T_1;x) &= (1+x)^2(1+2x)(1+6x+7x^2), \\ I(T_2;x) &= (1+x)(1+7x+14x^2+9x^3), \end{aligned}$$

where only $I(T_1;x)$ has all the roots real. It seems to be interesting to characterize (well-covered) trees whose independence polynomials have only real roots.

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