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Reliability of a system with renewable components and fast repair

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ABSTRACT. We present several results concerning on the ergodicity property of the birth and death processes with application in reliability theory, in other words we investigate the convergence in distribution of lifetime and repair time to a standard exponential random variable.

1. INTRODUCTORY REMARKS AND GENERAL RESULTS

In this paper we investigate the time to failure of a reparable system which has components with exponentially distributed lifetimes and exponentially distributed repair times. This make it possible to carry out the analysis in the framework of birth and death processes.

An important feature of almost all renewable systems considered in reliability theory is that the mean repair time of a component is many times smaller than the mean component lifetime. This "fast repair" property makes it possible to use powerful asymptotic methods in order to investigate the probabilistic behavior of system lifetime, when system behavior is described by a birth and death process. The presence of fast repair provides that the cumulative distribution function of system lifetime $\tau/E[T]$ is rather close to $1-e^{-x}$. This fact is very important for practical use because it allows us to evaluate system reliability only on the basis of the knowledge of $E[\tau]$.

Let $v(0)=0$. Let $\tau_{0,n}$ be the passage time from state 0 to state n, $n \geq 1$. An important formula is:

$$E[\tau_{0,n}] = \sum_{k=0}^{n-1} \frac{\sum_{i=0}^{k} \Theta_i}{\Theta_k \lambda_k} \tag{1.1}$$

Proposition 1.1. *If $F_n(t)$, $t \geq 0$ is cumulative distribution function a r.v. $\tau_{0,n}$, then*

$$F_n(t) = P(\tau_{0,n} \leq t) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{e^{zt}}{z\Delta_n(z)} dz \tag{1.2}$$

where $\Delta_n(z)$ is a polynomial

$$\Delta_n(z) = 1 + \Delta_{n,1} z + \Delta_{n,2} z^2 + ... + \Delta_{n,n} z^n \tag{1.3}$$

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Remark 1.1. The roots of $\Delta_n(z)$ have following properties:
(i) they are simple and negative;
(ii) between any two adjacent roots of $\Delta_n(z)$ there lies a root of $\Delta_{n-1}(z)$

Remark 1.2. The Laplace transform of the r.v. $\tau_{0,n}$ is:

$$E[e^{-z\tau_{0,n}}] = \frac{1}{\Delta_n(z)} \tag{1.4}$$

Remark 1.3. It follows from (1.4) that:

$$E[\tau_{0,n}] = \Delta_{n,1} \tag{1.5}$$

Proposition 1.2. *The Laplace transform of the r.v. $\tau_{0,n}/E[\tau_{0,n}]$ is:*

$$E\left[e^{-\frac{z\tau_{0,n}}{E[\tau_{0,n}]}}\right] = \frac{1}{1+z+a_2z^2+...+a_nz^n} \tag{1.6}$$

where the coefficient

$$a_2 = \frac{\Delta_{n,2}}{\Delta_{n,1}^2}$$

and

$$\Delta_{n,2} = \sum_{k=1}^{n-1} \frac{\sum_{s=1}^{k} E[\tau_{0,s}]\Theta_s}{\lambda_k\Theta_k}$$

2. Exponentially distributed lifetimes and repair times : fast repair

Let a system be with renewable standby components. The system has tow units in operation, one in a "warm" standby and two in a "cold" standby. Failed units are repaired in a repair shop which has two channels able to repair two failed units simultaneously. Any failed unit goes to repair, the place of a failed operating unit is taken by the unit from warm standby,the latter being replaced by a unit from cold standby. Repaired units return to the cold standby. System operation is illustrated by fig.1.

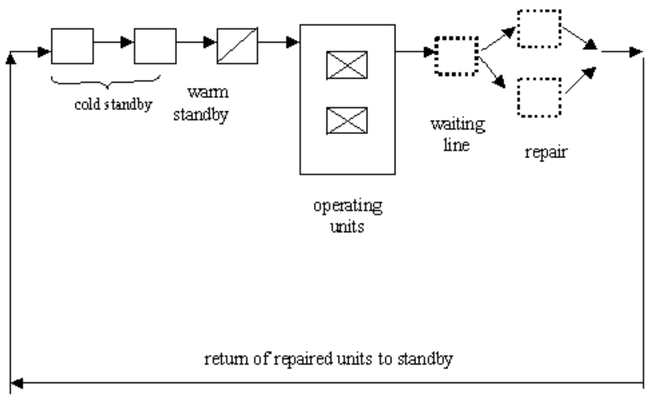

FIGURE 1. System with renewable standby

The system fails if the total number of failed units exceeds three; only one unit is available for operation. Failure rates are $\lambda^{(1)} = 1$, $\lambda^{(2)} = 0.6$ for the operating units, $\lambda^{(3)} = 0.4$ for the unit in warm standby and zero for cold standby. In other words, the lifetime of operating units are $\tau_1 \sim Exp(1)$, $\tau_2 \sim Exp(0.6)$ and the lifetime in warm standby is $\tau_3 \sim Exp(0.4)$. The repair rate in a channel is $\mu = 10$ with the repair time $\mu \sim Exp(10)$. The BD process are the "upward" transition rates $\lambda_0 = \lambda^{(1)} + \lambda^{(2)} + \lambda^{(3)} = 2$, $\lambda^{(1)} = \lambda^{(2)} = 2$, $\lambda^{(3)} = 1.6$ and the "downward" transition rates $\mu_1 = 10$, $\mu_2 = \mu_3 = 20$. The fast repair is reflected in the fact that the downward transition rates are much greater than the upward transition rates. Let us compute the mean transition time $E[\tau_{0,4}]$, which is the mean time to system failure in our example. The mean transition times are computed according to [4]. In our example, $\Theta_0 = 1, \Theta_1 = \lambda_0/\mu_1 = 0.2, \Theta_2 = \lambda_0\lambda_1/\mu_1\mu_2 = 0.02$, $\Theta_3 = \lambda_0\lambda_1\lambda_2/\mu_1\mu_2\mu_3 = 0.002$. It follows that:

$$E[\tau_{0,4}] = \frac{\Theta_0}{\Theta_0\lambda_0} + \frac{\Theta_0 + \Theta_1}{\Theta_1\lambda_1} + ... = 415.88$$

Because of the fast repair, $v(t)$ will have many returns to state zero before it hits for the first time the failure state $v(t) = 4$. The system starting with $v(0) = 0$ terminates with failure state with some probability p. This situation resembles the model of geometric distribution approximated by the exponential distribution (see [1], section 2.1). If the hypothesis about exponentiality is true, then $P(\tau > t) \approx e^{-t/E[\tau_{0,4}]}$.
For example, for $t = 50$, $P(\tau > 50) \approx e^{-50/415.88} = 0.8867$.
It will be shown later that this estimate is surprisingly accurate.
Now let us return to the birth and death process and assume that the system fails if $v(t)$ enters state n:

$$\tau_{0,n} = \inf\{t : v(t) = n \mid v(0) = 0\} \tag{2.7}$$

Theorem 2.1. *Let* $\xi = \tau_{0,n}/E[\tau_{0,n}]$. *If* $E[e^{-z\xi}] = \left(1 + z + a_2z^2 + ... + a_nz^n\right)^{-1}$, $a_2 < \frac{1}{4}$ *then*

$$\sup_{x\geq 0} \mid P(\xi > x) - e^{-x} \mid < \frac{1-\sqrt{1-4a_2}}{1+\sqrt{1-4a_2}} \tag{2.8}$$

Proof. We consider the following representation of the polynomial:

$$1 + z + a_2z^2 + ... + a_nz^n = \prod_{k=1}^{n}(1 + \alpha_k z) \tag{2.9}$$

By remark 1.1, all roots of the polynomial (2.9) are simple and nenegative; α_k are positive and distinct. In (2.9), if $n = 2$, then by the method of coefficients identification we have:

$$\begin{cases} \alpha_1 + \alpha_2 & = 1 \\ \alpha_1\alpha_2 & = a_2 \end{cases}$$

For $n = 3$:

$$\begin{cases} \alpha_1 + \alpha_2 + \alpha_3 & = 1 \\ \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3 & = a_2 \end{cases}$$

Generalisation:

$$\begin{cases} \alpha_1+\alpha_2+...+\alpha_n = 1 \\ a_2 = \sum_{i<j}\alpha_i\alpha_j = \frac{\left(\sum_{i=1}^{n}\alpha_i\right)^2 - \sum_{i=1}^{n}\alpha_i^2}{2} = \frac{1-\sum_{i=1}^{n}\alpha_i^2}{2} \end{cases}$$

We obtain the relation:

$$\alpha_1^2+\alpha_2^2+...+\alpha_n^2 = 1-2a_2 \tag{2.10}$$

Let $\alpha_1 > \alpha_2 > ... > \alpha_n$ and $\alpha = 1-\alpha_1$. How $a_2 < \frac{1}{4} \Rightarrow 1-2a_2 > \frac{1}{2}$. Further, $1-2a_2 < \alpha_1\alpha_1+\alpha_1\alpha_2+...+\alpha_1\alpha_n = \alpha_1(\alpha_1+\alpha_2+...+\alpha_n = \alpha_1 = 1-\alpha$.
We have the following inequality: $\frac{1}{2} < 1-\alpha \Rightarrow \alpha < \frac{1}{2}$.
Further, we obtain another inequality:

$$1-2a_2 = \alpha_1^2+\alpha_2^2+...+\alpha_n^2 < \alpha_1^2+(\alpha_1+\alpha_2+...+\alpha_n)^2 = (1-\alpha)^2+\alpha^2$$

thus

$$\alpha^2-\alpha+a_2 > 0 \tag{2.11}$$

The equation $\alpha^2-\alpha+a_2 = 0$ has two positive roots and $\Delta = 1-4a_2 > 0$. Together with $\alpha < \frac{1}{2}$ and the graph by fig. 2, the inequality (2.11) is true if:

$$\alpha < \frac{1-\sqrt{1-4a_2}}{2} \tag{2.12}$$

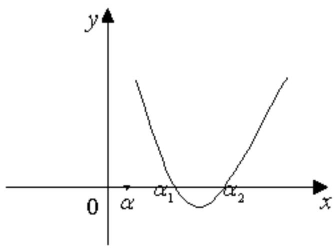

FIGURE 2. The graph of the function $f(x) = x^2-x+a_2$

Let us establish the following inequality: if $0<\alpha<1$ and $e^{\alpha x} \geq \alpha$, $\forall x > 0$:

$$\Psi(x) = e^{-\frac{x}{\alpha_1}} - e^{-x} \geq -\frac{1}{\alpha_1}+1 = -\frac{1-\alpha_1}{\alpha_1} = -\frac{\alpha}{1-\alpha} \tag{2.13}$$

We have that $\Psi(0) = 0$ and $\Psi(\infty) = 0$ and also $\Psi(x) < 0$, $\forall x > 0$. The equation $\Psi^{'}(x) = 0$ has a unique root x^* such that $x^*\left(1-\frac{1}{\alpha_1}\right) = \ln\alpha_1$.
Then:
$\Psi^*(x) = e^{-\frac{x^*}{\alpha_1}} - e^{-x^*} = -e^{-x^*}\left(1-e^{-\frac{x^*}{\alpha_1}+x^*}\right) = -e^{-x^*}\left[1-e^{x^*\ 1-\frac{1}{\alpha_1}}\right]$
$= -e^{-x^*}(1-\alpha_1) > -\frac{1-\alpha_1}{\alpha_1}$

Therefore,

$$\Psi(x) \geq \Psi^*(x) > -\frac{1-\alpha_1}{\alpha_1} = -\frac{\alpha}{1-\alpha}$$

The relationship

$$E\left[e^{-z\xi}\right] = \prod_{k=1}^{n} (1+\alpha_k z)^{-1}$$

suggests that

$$\xi = \eta_1 + \eta_2 + ... + \eta_n$$

where η_i are independent r.v., $E[\eta_i] = \alpha_i$, where $\eta_i \sim Exp(\alpha_i^{-1})$.
We have that:

$$E\left[e^{-z \sum_{i=1}^{n} \eta_i}\right] = E\left[\prod_{i=1}^{n} e^{-z\eta_i}\right] = \prod_{i=1}^{n} E\left[e^{-z\eta_i}\right]$$

How $E\left[e^{-z\xi}\right] = \prod_{k=1}^{n}(1+\alpha_k z)^{-1}$, we then have $E\left[e^{-z\eta_i}\right] = (1+\alpha_i z)^{-1}$.

Further,

$$\begin{aligned} E\left[e^{-z\eta_i}\right] &= \int_0^\infty e^{-zt} dF(t) dt = \int_0^\infty e^{-zt} d\left(1 - e^{-\frac{t}{\alpha_i}}\right) = \frac{1}{\alpha_i}\int_0^\infty e^{-\left(z+\frac{1}{\alpha_i}\right)t} dt \\ &= \frac{1}{\alpha_i}\frac{1}{z+\frac{1}{\alpha_i}} = \frac{1}{z\alpha_i + 1} \overset{(4)}{=} \frac{1}{\Delta_i(z)} \end{aligned}$$

results $\Delta_i(z) = 1 + \alpha_i z$ and in accordance with the relationship (1.3) and (1.5) $E(\eta_i) = \alpha_i$.

The event $\{\xi > x\}$ can occur in two ways: either $\eta_1 > x$ or $\eta_1 = y$, $y \in (0, x)$ and $\eta_2 + \eta_3 + ... + \eta_n > x - y$.

Then:

$$\begin{aligned} \delta(x) &= P(\xi > x) - e^{-x} = P(\eta_1 > x) + P(\eta_1 = y)P(\eta_2 + \eta_3 + ... + \eta_n > x - y) - e^{-x} \\ &= e^{-\frac{x}{\alpha_1}} - e^{-x} + \int_0^x P(\eta_2 + \eta_3 + ... + \eta_n > x - y) d\left(1 - e^{-\frac{y}{\alpha_1}}\right) \\ &= e^{-\frac{x}{\alpha_1}} - e^{-x} + \frac{1}{\alpha_1}\int_0^x e^{-\frac{y}{\alpha_1}} P(\eta_2 + \eta_3 + ... + \eta_n > x - y) dy \\ &\overset{x-y=t}{=} \underbrace{e^{-\frac{x}{\alpha_1}} - e^{-x}}_{-} + \frac{1}{\alpha_1}\int_0^x e^{-\frac{x-t}{\alpha_1}} P(\eta_2 + \eta_3 + ... + \eta_n > t) dt \\ &\leq \frac{1}{\alpha_1}\int_0^x P(\eta_2 + \eta_3 + ... + \eta_n > t) dt \leq \frac{1}{\alpha_1}\int_0^\infty P(\eta_2 + \eta_3 + ... + \eta_n > t) dt \\ &= \frac{1}{\alpha_1} E[\eta_2 + \eta_3 + ... + \eta_n] = \frac{1}{\alpha_1}\left[E(\eta_2) + E(\eta_3) + ... + E(\eta_n)\right] \\ &= \frac{\alpha_2 + \alpha_3 + ... + \alpha_n}{\alpha_1} = \frac{1-\alpha_1}{\alpha_1} = \frac{\alpha}{1-\alpha} \end{aligned}$$

Thus:

$$\delta(x) \leq \frac{\alpha}{1-\alpha} \tag{2.14}$$

After (2.13) and (2.14) results the following inequality:

$$\delta(x) \geq e^{-\frac{x}{\alpha_1}} - e^{-x} \geq -\frac{\alpha}{1-\alpha} \tag{2.15}$$

In view of (2.14), (2.15) and (2.12), we obtain:

$$\mid \delta(x) \mid \leq \frac{\alpha}{1-\alpha} < \frac{\frac{1-\sqrt{1-4a_2}}{2}}{1-\frac{1-\sqrt{1-4a_2}}{2}} = \frac{1-\sqrt{1-4a_2}}{1+\sqrt{1-4a_2}} \sim a_2 \text{ if } a_2 \to 0.$$

This proves (2.8).

Remark 2.4. The Theorem 2.1 establishes a remarkable fact: if $a_2 \to 0$, then $P(\xi > x) \overset{n\to\infty}{\to} e^{-x}$, that is r.v. $\xi = \dfrac{\tau_{0,n}}{E[\tau_{0,n}]} \overset{d}{\to} Exp(1)$

Remark 2.5. It is interesting to clarify the probabilistic meaning of the quantity a_2 which played the central role in proof of theorem 2.1. To show that $a_2 = 1 - \frac{E[\tau_{0,n}^2]}{2(E[\tau_{0,n}])^2}$. Theorem 2.1. is very useful for applications because it presents an upper bound on the deviation of $P(\xi > x)$ from e^{-x}. The formula for a_2 is presented in paragraph 1.

To apply the theorem 2.1 for the previous example. Let us estimate the difference between $P(\tau_{0,4} > 50) = P\left(\xi > \frac{50}{E[\tau_{0,4}]}\right)$ and its exponential approximation $e^{-\frac{50}{E[\tau_{0,4}]}}$. One obtains, by using the values of Θ_i and $E[\tau_{0,i}]$ computed earlier, that $\Delta_{n,2} = 78.88$. Then $a_2 = \frac{78.88}{(E[\tau_{0,4}])^2} = \frac{78.88}{415.88^2} = 0.00046$ and $|P(\tau_{0,4} > 50) - e^{-\frac{50}{E[\tau_{0,4}]}}| \leq 0.00046$ which means that the exponential approximation is indeed very accurate.

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