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On some relations on n-monoids

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ABSTRACT. In this paper we introduce the notion of n-monoid, $n > 2$, and we generalize for n-monoids two relations defined by F. Wehrung on the binary case. These relations can be interpreted as a "distance" between a n-monoid and the set of n-groups.

1. PRELIMINARY NOTIONS ABOUT n-MONOIDS

Wehrung [8] have studied the injective positively ordered monoids and defined two relations on the elements of a monoid, one preordering relation and other transitive and antisymmetric only. Later, Golan [4] have studied this relations for semirings with applications in computer science.

In this paper we define the notions of n-monoid, $n > 2$, positively ordered n-monoid and we generalize for n-monoids the relations defined by Wehrung [8]. The connections between the monoids and n-monoids, $n > 2$ and between the generalized relations we defined and Wehrung's are investigated, too.

Here we will often write a_i^j instead of $a_i, a_{i+1}, ..., a_j$, if $i \leq j$ and $\overset{(k)}{a}$ for $a, a, ..., a$ (k times). By a_i^j with $i > j$ we mean the empty sequence.

A set A together with an $n-$ary operation $()_+ : A^n \to A$ is called $n-semigroup$ if for any $k \in \{1, 2, ..., n\}$ and for all $a_1^{2n-1} \in A$ we have:

$$((a_1^n)_+, a_{n+1}^{2n-1})_+ = (a_1^k, (a_{k+1}^{k+n})_+,\ a_{k+n+1}^{2n-1})_+.$$

The sequence u_1^{n-1} is called the right (left) unit as system of elements if for all $x \in A$ holds $\left(x, u_1^{n-1}\right)_+ = x$ (respectively $\left(u_1^{n-1}, x\right)_+ = x$).

An $n-group$ [2] is an $n-$semigroup in which for all $a_1^n \in A$, the equations $(a_1^{i-1}, x, a_{i+1}^n)_+ = a_i;\ \ i \in \{1, 2, ..., n\}$ have an unique solution in A.

An $n-$semigroup (n-group) is called *semicommutative* if for all $a_1^n \in A$ we have $(a_1, a_2^{n-1}, a_n)_+ = (a_n, a_2^{n-1}, a_1)_+$ and *commutative* if $\forall\, \sigma \in S_n, (a_1^n)_+ = (a_{\sigma(1)}^{\sigma(n)})_+$. An n-semigroup is called *medial* if for all $a_{i1}^{in} \in A$; $i = \overline{1, n}$ the following relation holds

$$((a_{11}^{1n})_+, (a_{21}^{2n})_+, \ldots, (a_{n1}^{nn})_+)_+ = ((a_{11}^{n1})_+, (a_{12}^{n2})_+, \ldots, (a_{1n}^{nn})_+)_+ \,.$$

Any n-semigroup commutative is semicommutative and any n-semigroup semicommutative is medial [2].

An $n-$semigroup is called *cancellative* if for all

$$a, b, c_1^n \in A; (c_1^{i-1}, a, c_{i+1}^n)_+ = (c_1^{i-1}, b, c_{i+1}^n)_+ \Rightarrow a = b,\ i \in \{1, 2, ..., n\}$$

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Definition 1.1. An $n-$semigroup is called an $n-$*monoid* if there is at least one sequence $u_1^{n-1} \in A$ so called the unit of the n-monoid, such that

$$\forall\, x \in A,\ (x, u_1^{n-1})_+ = x = (u_{n-1}, u_1^{n-2}, x)_+$$

Therefore the sequence u_1^{n-1} is the right unit and $u_{n-1}u_1^{n-2}$ is left unit as system of elements. Because this unit is not necessary unique, let $U(A)$ be the set of units of the n-monoid $(A, ()_+)$. If in the sequel it is necessarily to specify the unit u_1^{n-1}, we denote the monoid by $\big(A, ()_+, u_1^{n-1}\big)$.

Remark 1.1. A semicommutative n-semigroup with a right unit is an n-monoid. Any n-group is an n-monoid.

Proposition 1.1. *If $u_1^{n-1} \in A$ is an unit of the n-monoid $(A, ()_+)$, then u_1^{n-1} is left unit and $u_{n-1}u_1^{n-2}$ is right unit too.*

Proof. Indeed, if u_1^{n-1} is an unit, i.e. u_1^{n-1} is right unit and $u_{n-1}u_1^{n-2}$ is a left unit of the n-monoid $(A, ()_+)$, then for all $x \in A$ we have

$$(u_1^{n-1}, x)_+ = \big(u_{n-1}u_1^{n-2}(u_1^{n-1}, x)_+\big)_+ = \big(u_{n-1}u_1^{n-3}(u_{n-2}u_1^{n-1})_+, x\big)_+ =$$
$$= (u_{n-1}u_1^{n-3}u_{n-2}, x)_+ = (u_{n-1}u_1^{n-2}, x)_+ = x,$$

i.e. u_1^{n-1} is a left unit.
Also

$$(x, u_{n-1}u_1^{n-2})_+ = ((x, u_{n-1}u_1^{n-2})_+ u_1^{n-1})_+ = (x, (u_{n-1}u_1^{n-2}u_1)_+, u_2^{n-1})_+ =$$
$$= (x, u_1, u_2^{n-1})_+ = (x, u_1^{n-1})_+ = x,$$

i.e. $u_{n-1}u_1^{n-2}$ is a right unit. □

Example

1. The set $\mathbb{N}$ respectively $\mathbb{Z}$ with the $(2n+1)$-ary operation, $n \in \mathbb{N}^a st$

$$(k_1, k_2, \dots, k_{2n+1})_+ = k_1 - k_2 + k_3 - \dots + k_{2n+1}$$

is a semicommutative $(2n+1)$-monoid with the units $\overset{(2n)}{a}$, $\forall a \in \mathbb{N}$, respectively $(2n+1)$-group.

2. The set $A = \mathbb{N} \times \mathbb{N}$ with ternary operation $(\ )_+ : A^3 \to A$,

$$((x_1, y_1), (x_2, y_2), (x_3, y_3))_+ = (x_1y_2x_3, y_1x_2y_3)$$

is a semicommutative 3-monoid with unit $(1,1)(1,1)$, while the set $B = \mathbb{Z} \times \mathbb{Z}$ together the above operation is a semicommutative 3-monoid with two units $(1,1)(1,1)$ and $(-1,-1)(-1,-1)$.

In [5] we have studied the connections between some n-ary algebraic structure and their corresponding binary structures.

If $(A, ()_+)$ is a n-semigroup and u_1^{n-2} are fixed elements in A and the binary operation $+ : A^2 \to A$ is defined by

$$x + y = (x, u_1^{n-2}, y)_+,$$

then $(A, +)$ is a semigroup [5]. This semigroup is called the binary reduced to respect to u_1^{n-2} of the n-semigroup $(A, ()_+)$. It is denoted by $\mathrm{red}_{u_1^{n-2}} A$.

As a consequence of this result, in the special case of n-monoids the following result holds.

Proposition 1.2. *If* $(A, ()_+, u_1^{n-1})$ *is a n-monoid then its binary reduced to respect to* u_1^{n-2}, $(A, +)$ *is a monoid with "zero",* u_{n-1}.

This monoid is denoted by $(\text{red}_{u_1^{n-2}} A, +, u_{n-1})$.

Proposition 1.3. *If the n-monoid* $(A, ()_+)$ *has more than one unit, let* u_1^{n-1} *and* v_1^{n-1} *be two units, then the binary reduces* $(red_{u_1^{n-2}} A, +, u_{n-1})$ *and* $(red_{v_1^{n-2}} A, *, v_{n-1})$ *are isomorphic.*

Proof. Indeed, if $u_1^{n-1}, v_1^{n-1}, u_{n-1}u_1^{n-2}, v_{n-1}v_1^{n-2}$ are left and right unit, then the map $f : A \to A$ defined by $f(x) = (v_{n-1}, x, u_1^{n-2})_+$ is an unitary homomorphism of the monoids $(\text{red}_{u_1^{n-2}} A, +, u_{n-1})$ and $(\text{red}_{v_1^{n-2}} A, *, v_{n-1})$. To see this we notice that

$$f(u_{n-1}) = (v_{n-1}, u_{n-1}, u_1^{n-2})_+ = v_{n-1}$$

and for all $x, y \in A$ we have

$$\begin{aligned} f(x + y) &= (v_{n-1}, x + y, u_1^{n-2})_+ = (v_{n-1}, (x, u_1^{n-2}, y)_+, u_1^{n-2})_+ = \\ &= (((v_{n-1}, x, u_1^{n-2})_+, v_1^{n-1})_+, y, u_1^{n-2})_+ = \\ &= ((v_{n-1}, x, u_1^{n-2})_+, v_1^{n-2}, (v_{n-1}, y, u_1^{n-2})_+)_+ = \\ &= (f(x), v_1^{n-2}, f(y))_+ = f(x) * f(y). \end{aligned}$$

Moreover f is one-to-one, because $f(x) = f(y)$ implies

$$(v_{n-1}, x, u_1^{n-2})_+ = (v_{n-1}, y, u_1^{n-2})_+$$

from where by Proposition 1.1 we have

$$\begin{aligned} &(v_1^{n-2}, (v_{n-1}, x, u_1^{n-2})_+, u_{n-1})_+ = (v_1^{n-2}, (v_{n-1}, y, u_1^{n-2})_+, u_{n-1})_+ \,, \quad \text{or} \\ &((v_1^{n-1}, x)_+, u_1^{n-1})_+ = ((v_1^{n-1}, y)_+, u_1^{n-1})_+ \,, \end{aligned}$$

that is $x = y$.

The map f is surjective, for any $y \in A$ there is $x = (v_1^{n-2}, y, u_{n-1})_+ \in A$ such that $f(x) = y$.

Therefore f is a monoids isomorphism. □

An other approache for reducing an n-semigroup to a semigroup is the following construction [5]:

Let $(A, ()_+)$ be an n-semigroup with right unit u_1^{n-1}. If on $G = \bigcup_{i=1}^{n-1} A^i$ is defined the relation

$$a_1^i \rho {a'}_1^i \Leftrightarrow \big(u_i^{n-1} a_1^i\big)_+ = \big(u_i^{n-1} {a'}_1^i\big)_+ \Leftrightarrow \big(x_1^{n-i} a_1^i\big)_+ = \big(x_1^{n-i} {a'}_1^i\big)_+ \,,$$

for all $x_1^i \in A$, then ρ is a equivalence relation.

Let $\big[a_1^i\big] = \rho\big(a_1^i\big)$. Obviously that $\big[u_1^{n-1}\big] = \big\{v_1^{n-1}, \big(x, v_1^{n-1}\big)_+ = x, \ \forall\, x \in A\big\}$, is the class of all right unit as system of elements. Moreover, if on G/ρ we define the binary operation:

$$\left[a_1^i\right]\cdot\left[b_1^j\right]=\begin{cases}\left[a_1^ib_1^j\right], & \text{if} \quad i+j<n\\ \left[a_1^{i+j-n}\left(a_{i+j-n+1}^ib_1^j\right)_+\right], & \text{if} \quad i+j\geq n,\end{cases}$$

then G/ρ is a monoid with unit $\left[u_1^{n-1}\right]$.

For all $a_1^n\in A$ we have $[(a_1\ldots a_n)_+]=[a_1]\cdot[a_2]\cdot\ldots\cdot[a_n]$.

The set $A_0=A^{n-1}/\rho$ is a submonoid of G/ρ.

Moreover, if $(A,(\,)_+)$ is a semicommutative n-group, then A_0 is a invariant commutative submonoid. To see this we notice that for all $\left[x_1^i\right]\in G/\rho$ and $\left[a_1^{n-1}\right]\in A_0$, we have

$$\begin{aligned}&\left[x_1^i\right]\cdot\left[a_1^{n-1}\right]=\left[x_1^{i-1}\left(x_ia_1^{n-1}\right)_+\right]=\left[x_1^{i-1}\left(a_{n-1}a_1^{n-2}x_i\right)_+\right]=\\&=\left[x_1^{i-2}a_{n-2}\left(a_{n-1}a_1^{n-3}x_{i-1}^i\right)_+\right]=\cdots=\left[a_{n-i}^{n-2},\left(a_{n-1}a_1^{n-i-1}x_1^i\right)_+\right]\\&=\left[a_{n-i}^{n-1}a_1^{n-i-1}\right]\cdot\left[x_1^i\right].\end{aligned}$$

The two methods of reducing the n-monoids to monoids are not independent:

Proposition 1.4. *If $(A,(\,)_+)$ is a n-semigroup with right unit u_1^{n-1}, then the reduced to respect to u_1^{n-2} of $(A,(\,)_+)$, $(red_{u_1^{n-2}}A,+,u_{n-1})$ is isomorphic to submonoid A_0 of G/ρ if and only if $u_{n-1}u_1^{n-2}$ is left unit in $(A,(\,)_+)$, that is $(A,(\,)_+)$ is a n-monoid.*

The proof for n-semigroup is given in [5].

This result justifies our definition for an n-monoid, as an n-semigroup for which there is at least one system of $(n-1)$ elements u_1^{n-1} such that it is right unit and $u_{n-1}u_1^{n-2}$ is a left unit.

2. Positively ordered n-monoids and some relations on the $n-$monoids

We generalize for n-monoid the notion of positively ordered monoid defined in [8]:

Definition 2.1. Positively ordered $n-$monoid (from now on P.O. $n-M$) is a structure

$$\mathbf{A}=\left(A,\ ()_+,U(A),\leq\right)$$

such that:

O_1. $(A,\ ()_+,U(A))$ is a semicommutative $n-$monoid;

O_2. $(A,\leq)$ is a partially preordered set under a relation "$\leq$ " and for every unit $u_1^{n-1}\in U(A)$ holds

$$u_{n-1}\leq a,\quad \text{for all}\quad a\in A.$$

O_3. If a,b are elements of A, then $a\leq b$ implies

$$(*)\qquad \left(c_1^{i-1},a,c_{i+1}^n\right)_+\leq\left(c_1^{i-1},b,c_{i+1}^n\right)_+,\quad \text{for all}\quad c_1^n\in A,\ i=1,2,...,n.$$

Proposition 2.1. *The condition O_3 is fulfilled if the relations (*) hold only for $i=1$ and $i=2$.*

Proof. Indeed, if $a \leq b$ implies $(a, c_2^n)_+ \leq (b, c_2^n)_+$ and $(c_1, a, c_3^n)_+ \leq (c_1, b, c_3^n)_+$ for all $c_1^n \in A$, then $(c_2, a, u_1^{n-2}) \leq (c_2, b, u_1^{n-2})_+$ and

$$(c_1, c_2, a, c_4^n)_+ = (c_1, c_2, (a, u_1^{n-1})_+, c_4^n)_+ = (c_1, (c_2, a, u_1^{n-2})_+, u_{n-1}, c_4^n)_+ \leq$$
$$\leq (c_1, (c_2, b, u_1^{n-2})_+, u_{n-1}, c_4^n)_+ = (c_1, c_2, (b, u_1^{n-1})_+, c_4^n)_+ = (c_1, c_2, b, c_4^n)_+ .$$

By induction if the relation (*) hold for $i = 1, 2, \dots, k-1$, then it holds for k too, $k = 3, \dots, n$. $\square$

Remark 2.1. Using the transitivity of the relation $\leq$, from O_3 it follows straight forward that if $a_i \leq b_i$ for all $i = \overline{1, n}$ we have $\left(a_{\sigma(1)}^{\sigma(n)}\right)_+ \leq \left(b_{\sigma(1)}^{\sigma(n)}\right)_+$ for arbitrary permutation σ of the set $\{1, 2, ..., n\}$.

Remark 2.2. If in the $n-$monoid $\left(A,\ ()_+, u_1^{n-1}\right)$ the cancellation laws hold, we immediately obtain a partial converse of O_3.

If $\left(c_1^{i-1}, a, c_{i+1}^n\right)_+ \leq \left(c_1^{i-1}, b, c_{i+1}^n\right)_+$ and $a \neq b$, then $a < b$, or a and b are incomparable.

Note that in the case $n = 2$, $\mathbf{A}$ is a positively ordered monoid in the sense of Wehrung.

Remark 2.3. If $(A,\ ()_+, U(A), \leq)$ is a P.O. n-M. and $u_1^{n-1} \in U(A)$, then $(\mathrm{red}_{u_1^{n-2}} A, +, u_{n-1}, \leq)$ is a positively ordered monoid.

If in Definition 2.1 we replace O_2 by

O_2'. $(A, \leq)$ is an ordered set under relation $\leq$,

then $\mathbf{A}$ is called a partially ordered n-monoid.

The notion of partially ordered n-groups is defined and studied by Crombez [1] and studied by Ušan [6], [7], too.

In what it follows we generalize in the case of the $n-$monoids some relations, defined by Wehrung [8] on the binary case.

Definition 2.2. Let $(\ A, ()_+, u_1^{n-1})$ be an $n-$monoid and $a, b \in A$. We define the following relations:

(1) $a \preccurlyeq b \Leftrightarrow (\exists\ x_1^{n-1} \in A)((a, x_1^{n-1})_+ = b)$;
(2) $a \ll b \Leftrightarrow (\exists\ x_1^{n-2} \in A)((a, x_1^{n-2}, b)_+ = b)$;
(3) $a \preccurlyeq_u b \Leftrightarrow (\exists\ x \in A)((a, u_1^{n-2}, x)_+ = b)$;
(4) $a \ll_u b \Leftrightarrow (a, u_1^{n-2}, b)_+ = b$.

Remark 2.4. The above relations is not independent. So

$$a \ll_u b \Rightarrow a \ll b \Rightarrow a \preccurlyeq b \Rightarrow a \preccurlyeq_u b \Rightarrow a \preccurlyeq b,$$

i.e. the two relations (1) and (3) are the same.

The first two implications are obviously. If $a \preccurlyeq b$, then there is $x_1^{n-1} \in A$ such that $\left(a, x_1^{n-1}\right)_+ = b$, hence

$$b = \left(\left(a, u_1^{n-1}\right)_+, x_1^{n-1}\right)_+ = \left(a, u_1^{n-2}, \left(u_{n-1}, x_1^{n-1}\right)_+\right)_+ \quad \text{and} \quad a \preccurlyeq_u b.$$

Conversely, if $a \preccurlyeq_u b$ there is $x \in A$ such that $\left(a, u_1^{n-2}, x\right)_+ = b$, hence $a \preccurlyeq b$. Therefore $a \preccurlyeq b \Longleftrightarrow a \preccurlyeq_u b$.

As a consequence the following holds

Remark 2.5. If v_1^{n-1} is other unit of the n-monoid $\left(A, ()_+, u_1^{n-1}\right)$, then

$$a \preccurlyeq_u b \Leftrightarrow a \preccurlyeq_v b.$$

Proposition 2.2. 1°. *If* $\left(A, ()_+, u_1^{n-1}\right)$ *is a semicommutative* $n-$*monoid, then* $\left(A,\ ()_+, u_1^{n-1}, \preccurlyeq\ \right)$ *is a P.O.* $n-M$.

We will call the relation $\preccurlyeq$ the minimal or canonical preordering.

2°. *The relation* (2) " $\ll$ " *is not necessarily reflexive but it is transitive and compatible with the* n*-ary operation.*

If for all $a \in A$ *there are* $x_1^{n-2} \in A$; $\left(a, x_1^{n-2}, a\right)_+ = a$, *then* $\left(A, ()_+, u_1^{n-1}, \ll\ \right)$ *is P.O.* $n-M$.

3°. *A necessary and sufficient condition for that the relation* (4) $''\ll_u$ " *to be a partial ordered is*

$$\forall a \in A,\ \left(a,\ u_1^{n-2}\ a\right)_+ = a.$$

Proof. 1° O_1. From $\left(a\ u_1^{n-1}\right)_+ = a$, $\forall a \in A$, we have $a \preccurlyeq a$, i.e., the relation is reflexive. This relation is transitive too: $a \preccurlyeq b$ and $b \preccurlyeq c$ implies $\exists\, x_1^{n-1} \in A$; $\left(a, x_1^{n-1}\right)_+ = b$ and $\exists\, y_1^{n-1} \in A$; $\left(b, y_1^{n-1}\right)_+ = c$, hence $\left(ax_1^{n-2}(x_{n-1}, y_1^{n-1})_+\right)_+ = c$, at where $a \preccurlyeq c$. Moreover:

O_2. For any $a \in A$, $\left(u_{n-1}u_1^{n-2}a\right)_+ = a$ implies $u_{n-1} \preccurlyeq a$;

O_3. If $a \preccurlyeq b$, then $\exists\, x_1^{n-1} \in A$ such that $\left(a, x_1^{n-1}\right)_+ = b$.

Then for all $c_1^n \in A$ and $i = \overline{1,n}$ by semicommutativity and associativity of $n-$ary operation we obtain:

$$\left(c_1^{i-1}, b, c_{i+1}^n\right)_+ = \left(c_1^{i-1}, (a, x_1^{n-1})_+, c_{i+1}^n\right)_+ = \left(\left(c_1^{i-1}, a, c_{i+1}^n\right)_+, x_{n-i+1}^n x_1^{n-i}\right)_+.$$

From this result $\left(c_1^{i-1}, a, c_{i+1}^n\right)_+ \preccurlyeq \left(c_1^{i-1}, b, c_{i+1}^n\right)_+$.

2°. The transitivity is immediately. Let now $a_i \ll b_i$ be $i = \overline{1,n}$, i.e. there are $x_{i1}^{i,n-2} \in A$, $i = \overline{1,n}$, such that $(a_i, x_{i1}^{i,n-2}, b_i)_+ = b_i$. Because the n-ary operation is semicommutative it is medial and

$$\begin{aligned}(b_1^n)_+ = ((a_1, x_{11}^{1,n-2}, b_1)_+, \ldots, ((a_n, x_{n1}^{n,n-2}, b_n)_+)_+) = \\ ((a_1^n)_+, (x_{11}^{n1})_+, \ldots, (x_{1,n-2}^{n,n-2})_+, (b_1^n)_+)_+\end{aligned}$$

whence $(a_1^n)_+ \ll (b_1^n)_+$.

The relation (2) is not necessarily reflexive. If for

$$\forall\, a \in A\ \exists\, x_1^{n-2} \in A;\ \left(a, x_1^{n-2}, a\right)_+ = a,$$

then $\left(A, ()_+, u_1^{n-1}, \ll\ \right)$ is a P.O.$n-M$.

3°. If $a \ll_u b$ and $b \ll_u a$, we have $\left(a\, u_1^{n-2} b\right)_+ = b$ and $\left(b\, u_1^{n-2} a\right)_+ = a$. By semicommutivity of n-ary operation we have $a = b$. From $a \ll_u b$ and $b \ll_u c$ result

$$c = \left(b\, u_1^{n-2} c\right) = \left(\left(a\, u_1^{n-2} b\right)_+ u_1^{n-2} c\right)_+ = \left(a\, u_1^{n-2}\left(b\, u^{n-2} c\right)_+\right)_+ = \left(a\, u_1^{n-2} c\right)_+,$$

hence $a \ll_u c$. Then the relation (4) is an orderer relation if and only if for all $a \in A$ we have $\left(a\, u_1^{n-2} a\right)_+ = a$. □

We remark that the relation $\preccurlyeq$ is a preorder but not necessarily a partial order on A. A sufficient condition for it to be a partial order is that $((a\,u_1^{n-2},x)_+u_1^{n-2},y)_+=a$ imply $(a\,u_1^{n-2},x)_+=a$, for all $a,x,y\in A$. If the relation $\preccurlyeq$ is a partial order on A, then analogous binary case we say that the n-monoid A is difference ordered. The 3-monoid, $(A,()_+)$ from example 2, has this property, but the 3-monoid $(B,()_+)$ has not this property. Of course, if $a\preccurlyeq b$, then the elements $x_1^{n-1}\in A$ satisfying $(a\,x_1^{n-1})_+=b$ need not be unique.

The following statements generalize the results of F. Wehrung [7] from the binary in the n-ary, $n\geq 2$, case.

Proposition 2.3. *Let* $\big(A,()_+,u_1^{n-1},\leq\big)$ *be a P.O.*$n-M$*. Then*

(*i*) *If* **A** *is minimal, then* $a\ll b$ *and* $b\preccurlyeq c$ *implies* $a\ll c$*;*

(*ii*) *If the relation "*$\leq$*" is antisymmetric, then* $a\leq b$ *and* $b\ll_u c$ *implies* $a\ll_u c$*.*

Proof. (i) The relation $b\preccurlyeq c$ implies that there are $x_1^{n-1}\in A$ such that $\big(b,x_1^{n-1}\big)_+=c$. From $a\ll b$ that are $y_1^{n-2}\in A$ such that $(ay_1^{n-2}b)_+=b$. Then

$$c=\big(bx_1^{n-1}\big)_+=\Big(\big(ay_1^{n-2}b\big)_+x_1^{n-1}\Big)=\Big(ay_1^{n-2}\big(bx_1^{n-1}\big)_+\Big)_+=\big(ay_1^{n-2}c\big)$$

and $a\ll c$.

(ii) Because $\big(b,u_1^{n-2},c\big)_+=c$ and **A** is P.O. $n-M$, the relation $a\leq b$ implies $\big(a,u_1^{n-2},c\big)_+\leq\big(b,u_1^{n-2},c\big)_+$, hence $\big(a,u_1^{n-2},c\big)_+\leq c$. But, by reflexivity of the relation $\leq$, from $u_{n-1}\leq a$; $c\leq c$; $u_i\leq u_i$; $i=\overline{1,n-2}$ we have

$$c=\big(u_{n-1}u_1^{n-2}c\big)_+\leq\big(a\,u_1^{n-2}c\big)_+.$$

By antisymmetry of relation "$\leq$" results $\big(a,u_1^{n-2},c\big)_+=c$, therefore $a\ll_u c$. $\square$

Furthermore, if A is minimal P.O.$n-M$, then this last property (ii) characterizes antisymmetry.

Proposition 2.4. *The following condition on a semicommutative* $n-$*monoid with unit* u_1^{n-1} *are equivalent*

(*i*) *If* $a\preccurlyeq b$ *and* $b\ll_u c$ *in* A*, then* $a\ll_u c$*;*

(*ii*) *If* $a\preccurlyeq b$ *and* $b\preccurlyeq a$ *in* A*, then* $a=b$*.*

Proof. Assume (i) and let $a\preccurlyeq_u b$ and $b\preccurlyeq_u a$. Then there exist elements x and y in A satisfying $\big(au_1^{n-2}x\big)_+=b$ and $\big(bu_1^{n-2}y\big)_+=a$.

Thus

$$\begin{aligned}a&=((au_1^{n-2}x)_+u_1^{n-2}y)_+=((xu_1^{n-2}a)_+u_1^{n-2}y)_+=\\&=(xu_1^{n-2},(a,u_1^{n-2},y)_+)_+=(xu_1^{n-2},(y,u_1^{n-2},a)_+)_+=\\&=((x,u_1^{n-2},y)_+,u_1^{n-2},a)_+\end{aligned}$$

so $(x,u_1^{n-2},y)_+\ll_u a$. But $x\preccurlyeq_u\big(xu_1^{n-2}y\big)_+$ and by (i), this implies that $x\ll_u a$. Therefore $a=\big(a\,u_1^{n-2}x\big)_+=b$.

Now, conversely, assume (ii) and let $a\preccurlyeq_u b$ and $b\ll_u c$ in A. Then there exists an element x of A satisfying $\big(au_1^{n-2}x\big)_+=b$ so,

$$((au_1^{n-2}c)_+u_1^{n-2},x)_+=((au_1^{n-2}x)_+u_1^{n-2}c)_+=(bu_1^{n-2}c)_+=c,$$

proving that $\left(au_1^{n-2}c\right) \preccurlyeq c$. But $c \preccurlyeq \left(au_1^{n-2}c\right)_+$ and so, by (ii), we have $c = \left(au_1^{n-2}c\right)_+$, proving that $a \ll c$. $\square$

Remark 2.6. If $(A, ()_+)$ is an $n-$group, then the relations (1) and (2) are the coarse preordering $A \times A$.

The connection between the above relations and Wehrung's is given by

Proposition 2.5. *If $(A, ()_+, u_1^{n-1})$ is an semicommutative n-monoid, $a, b \in A$, then in the $(red_{u_1^{n-2}}A, +, u_{n-1})$ we have:*

$$a \preccurlyeq b \Leftrightarrow (\exists\, c \in A);\ a + c = b$$
$$a \ll_u b \Leftrightarrow a + b = b.$$

The following proposition describes so-called the right and left cone (cf. Fuchs [3]), i.e. the set $K_r(a) \overset{\text{def}}{=} \{x; a \leq x\}$, respectively $K_l(a) \overset{\text{def}}{=} \{x; x \leq a\}$. For example, if $(A, ()_+, u_{n-1}, \leq)$ is an positively ordered n-monoid then $K_r(u_{n-1}) = A$ and $\{v_{n-1}; v_1^{n-1} \in U(A)\} \subseteq K_l(u_{n-1})$.

Proposition 2.6. *Let $\left(A, ()_+, \leq, u_1^{n-1}\right)$ be a positively ordered n-monoid. Then $(K_r(a), ()_+)$ and $(K_l(a), ()_+)$ are sub-n-semigroups of the n-monoid $(A, ()_+)$ if and only if $a \leq (\overset{(n)}{a})_+$ respectively $(\overset{(n)}{a})_+ \leq a$.*

Proof. Let $a \leq (\overset{(n)}{a})_+$ be. For every sequence $x_1^n \in K_r(a)$ we have $a \leq x_i$; $i = \overline{1, n}$. Hence, by Remark 2.1 we conclude that $a \leq (\overset{(n)}{a})_+ \leq (x_1^n)_+$. Whence, by transitivity of $\leq$, we have $a \leq (x_1^n)_+$, i.e. $(x_1^n)_+ \in K_r(a)$. So $K_r(a)$ is a sub-n-semigroup of the n-monoid $(A, ()_+)$.

Assume now that $K_r(a)$ is a sub-n-semigroup pf $(A, ()_+)$. Because $a \in K_r(a)$, it follows that $(\overset{(n)}{a})_+ \in K_r(a)$, whence $a \leq (\overset{(n)}{a})_+$. $\square$

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