CARPATHIAN J. MATH.
20 (2004), No. 1, 95-100

A quadratic programming method for saddle point formulations in contact problems with friction

NICOLAE POP and IOANA ZELINA

ABSTRACT. The paper is concerned with the numerical solution of the quasi-variational inequality modelling a contact problem with Coulomb friction. After discretization of the problem by mixed finite elements and with Lagrangian formulation of the problem by choosing appropriate multipliers, the duality approach is improved by splitting the normal and tangential stresses. The novelty of our approach in the present paper consists in the splitting of the normal stress and tangential stress, which leads to a better convergence of the solution, due to a better conditioned stiffness matrix. This better conditioned matrix is based on the fact that these blocks diagonal matrices obtained, contain coefficients of the same size order. For the saddle point formulation of the problem, using static condensation, we obtain a quadratic programming problem. We use Gauss-Seidel iterations to approximate the solution of this problem.

1. VARIATIONAL FORMULATIONS OF CONTACT PROBLEMS WITH COULOMB FRICTION

Consider a linear elastic body that occupies a domain $\Omega \subset \mathbb{R}^2$, with a Lipschitz boundary Γ. Let Γ_0, Γ_1 and Γ_C be open and disjoint parts of Γ which do not depend on time such that $\Gamma = \overline{\Gamma}_0 \cup \overline{\Gamma}_1 \cup \overline{\Gamma}_C$.

Assume that the body is subjected to volume forces of density $F \in (L^2(\Omega))^2$, to surface traction of density $T \in (L^2(\Gamma_1))^2$ and is held fixed on Γ_0. The Γ_C denotes a contact part of boundary where unilateral contact and Coulomb friction conditions between Ω and perfectly rigid foundation are considered. Supposing that a positive coefficient Φ of Coulomb friction is given, we introduce the space of *virtual displacements*

$$V = \left\{ v \in (H^1(\Omega))^2 | v = 0 \text{ on } \Gamma_0 \right\}$$

and its convex subset of *kinematically admissible displacements*

$$K = \{ v \in V | v_n \equiv v \cdot n \le d \text{ on } \Gamma_C \}.$$

Here, $d \in C(\bar{\Gamma}_C)$, $d \ge 0$ is an initial gap between the body and the rigid foundation and $n \in (L^\infty(\Gamma_C))^2$ denotes the outer unit normal vector to boundary Γ.

We assume that the normal force on Γ_C is known so that one can evaluate the non-negative slip bound $g \in L^\infty(\Gamma_C)$ as a product of the friction coefficient and the normal stress, i.e. $g = \Phi\lambda_1$, when λ_1 is the normal count stress.

Received: 15.10.2004; In revised form: 10.01.2005
2000 *Mathematics Subject Classification.* 35J85, 49J35, 65N30, 74M15, 74S05, 65Y20.
Key words and phrases. *Contact problem with Coulomb friction, dual mixed formulation, mixed finite element, saddle point problem, quadratic programming, Schur complement.*

The *primal variational formulation* of the contact problem with given friction is:

(P_1) *Find* $u \in K$ *such that* $J(u) = \min_{v \in K} J(v)$.

The minimized functional representing the total potential energy of the body has the form:

$$J(v) = \frac{1}{2}a(v,v) - L(v) + j(v)$$

where:

- the bilinear form
$$a(v,w) = \int_\Omega a_{ijkl}\varepsilon_{ij}(v)\varepsilon_{kl}(w)dx$$
contains the fourth order symmetric tensor a_{ijkl}, $i,j,k,l = 1,2$, representing the Hook's low $\sigma_{ij}(v) = a_{ijkl}\varepsilon_{kl}(v)$ and linearized strain tensor $\varepsilon(v) = \frac{1}{2}(\nabla v + \nabla^T v)$;
- linear functional L is given by:
$$L(v) = \int_\Omega F v dx + \int_{\Gamma_1} T v ds;$$
- the sublinear functional $\overline{j}$ is given by:
$$\overline{j}(v) = \int_{\Gamma_C} g\,|v_t|\,ds$$
where $v_t \equiv v \cdot t$ and $t \in (L^\infty(\Gamma_C))^2$ denotes the unit tangent vector to boundary Γ.

It is known that (P_1) is non-differentiable due to the sublinear term j, and has a unique solution [3].

The primal variational formulation (P_1) is equivalent to the quasi-variationl inequality:

(P_2) *Find* $u \in K$ *such that* $a(u, v-u) + \overline{j}(v-u) \geq (L, u-v) \ \ \forall v \in K.$

The existence and uniqueness of the solution of this quasi-variational inequality are proven under the assumption the Φ is sufficiently small and $mes(\Gamma_0) > 0$ [1].

The *Lagrangian formulation* of the problem (P_1) is given by:

$L : V \times \Lambda_1 \times \Lambda_2 \to$R, and

$$L(v,\mu_1,\mu_2) = \frac{1}{2}a(v,v) - L(v) + \langle \mu_1, v_n - d\rangle + \int_{\Gamma_C} \mu_2 v_t ds$$

where $\Lambda_1 = \{\mu_1 \in H^{-\frac{1}{2}}(\Gamma_C) | \mu_1 \geq 0\}$, $\Lambda_2 = \{\mu_2 \in L^\infty(\Gamma_C) | \ \ |\mu_2| \leq g \text{ on } \Gamma_C\}$.

The space $H^{-\frac{1}{2}}(\Gamma_C)$ is the dual of

$$H^{\frac{1}{2}}(\Gamma_C) = \{h \in L^2(\Gamma_C) | \, \exists v \in V \ s.t. \ h = v_n \text{ on } \Gamma_C\}$$

and the ordering $\mu_1 \geq 0$ means, in the variational form, that: $\langle \mu_1, v_n - d\rangle \leq 0$, $\forall\, v \in K$, where $\langle \cdot,\cdot\rangle$ denotes the duality pairing between $H^{-\frac{1}{2}}(\Gamma_C)$ and $H^{\frac{1}{2}}(\Gamma_C)$. Since $L^2(\Gamma_C)$ is dense in $H^{-\frac{1}{2}}(\Gamma_C)$, the duality pairing $\langle \cdot,\cdot\rangle$ is represented by the scalar product in $L^2(\Gamma_C)$.

The Lagrange multipliers μ_1, μ_2 are considered as functional on the part of the boundary Γ. It is important that the Lagrange multipliers do have mechanical significance: while the first one counts for the non-penetration conditions and represents the normal stress, the second one removes the non-differentiability of the sublinear functional

$$\bar{j}(v) = \sup_{\mu_2 \in \Lambda_2} \int_{\Gamma_C} \mu_2 v_t ds$$

and represents the tangential stress.

The equivalence between the problem (P_1) and the lagrangian formulation is given by:

$$\inf_{v \in K} J(v) = \inf_{v \in V} \sup_{\mu_1 \in \Lambda_1, \mu_2 \in \Lambda_2} L(v, \mu_1, \mu_2).$$

By the mixed variational formulation of the problem (P_1) we mean a saddle point problem:

$$\textit{find } (w, \lambda_1, \lambda_2) \in V \times \Lambda_1 \times \Lambda_2 \textit{ such that}$$

$$\text{(P3)}\ L(w, \mu_1, \mu_2) \le L(w, \lambda_1, \lambda_2) \le L(v, \lambda_1, \lambda_2), \quad \forall (v, \mu_1, \mu_2) \in V \times \Lambda_1 \times \Lambda_2.$$

It is known that (P_3) has a unique solution [7] and its first component $w = u \in K$ solves (P_1) and the Lagrange multipliers λ_1, λ_2 represent the normal and tangential contact stress on the contact part of the boundary, respectively.

Remarks.

1^o. For the contact problem with Coulomb friction, we use the formula $g \equiv \Phi\lambda_1$, for the slip bound on the contact boundary Γ_C , where $\lambda_1 \equiv \lambda_1(g)$ is the normal stress on Γ_C and Φ is the coefficient of friction. Unfortunately this problem cannot be solved as a convex quadratic programming problem because g is an a priori parameter in (P_3), while λ_1 is an a posteriori one.

2^o. Because we can consider the mapping $\Psi : \Lambda_1 \to \Lambda_1,\ \Psi : g \to \lambda_1 \equiv \lambda_1(g)$ defined by the second component of the solution for the contact problem with given friction (P_3), the solution of the contact problem with Coulomb friction will be defined as a fixed point of this maping in Λ_1. Results concerning the existence of fixed points for sufficiently small friction coefficients may be found in [5].

2. Finite element approximations of the contact problems with Coulomb friction

We suppose that Ω is a polygonal domain with regular triangulation T_h that is consistent with the decomposition of boundary $\Gamma = \overline{\Gamma}_0 \cup \overline{\Gamma}_1 \cup \overline{\Gamma}_C$. On T_h we consider the classical piecewise linear basis functions $\{\varphi_j\}$ defining the finite element subspace $V_h \subset V$ with $dim V_h = n$. We denote by m the number of contact nodes on $\overline{\Gamma}_C$. On Γ_C we construct a regular partition T_H with the norm denoted by H independent of the triangulation T_h. On T_H we consider the space Λ_H of piecewise constant functions with $dim\Lambda_H = p$, and we define the spaces $\Lambda_{i,H} = \Lambda_i \cap \Lambda_H$ for $i = 1, 2$.

With this discretization, if we replace the space V by the space V_h , the problem (P_1) becomes:

$(P_1)^h$ *Find* $\mathbf{u} \in \mathbf{K_h}$ *such that* $\mathbf{J}(\mathbf{u}) = \min_{\mathbf{v} \in \mathbf{K_h}} \mathbf{J}(\mathbf{v})$,

where $J(v) = \frac{1}{2}\boldsymbol{v}^T\boldsymbol{K}\boldsymbol{v} - \boldsymbol{v}^T\boldsymbol{f} + \boldsymbol{g}^T|\boldsymbol{T}\boldsymbol{v}|$ and $K_h = \{\boldsymbol{v} \in \mathbb{R}^n | \boldsymbol{N}\boldsymbol{v} \le \boldsymbol{d}\}$. Here, we by denote $\boldsymbol{K} \in \mathbb{R}^{n\times n}$ the positive definite stiffness matrix, $\boldsymbol{f} \in \mathbb{R}^n$ is the load vector, $\boldsymbol{g} \in \mathbb{R}^m$ is the nodal slip bounds vector for contact nodes. The matrices $\boldsymbol{N}, \boldsymbol{T} \in \mathbb{R}^{m\times n}$ contain the rows of the normal and tangential vectors in the contact nodes, respectively, and $\boldsymbol{d} \in \mathbb{R}^n$ is the vector of distances between the contact nodes and the rigid foundation.

The *matrix form of the Lagrangian* for the problem $(\mathrm{P}_1)^h$ is:

$$L(\boldsymbol{v}, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2) = \frac{1}{2}\boldsymbol{v}^T\boldsymbol{K}\mathbf{v} - \boldsymbol{f}^T\mathbf{v} + \boldsymbol{\mu}_2^T\boldsymbol{T}\boldsymbol{v} + \boldsymbol{\mu}_1^T(\boldsymbol{N}\boldsymbol{V} - \mathbf{d})$$

where $\boldsymbol{\mu}_1 \in \boldsymbol{\Lambda}_1$, $\boldsymbol{\mu}_2 \in \boldsymbol{\Lambda}_2$ are the Lagrange multipliers and $\boldsymbol{\Lambda}_1 = \{\boldsymbol{\mu}_1 \in \mathbb{R}^m | \boldsymbol{\mu}_1 \geq \mathbf{0}\}$, $\boldsymbol{\Lambda}_2 = \{\boldsymbol{\mu}_2 \in \mathbb{R}^m || \boldsymbol{\mu}_2| \leq \boldsymbol{g}\}$.

The algebraic mixed formulation of $(\mathrm{P}_1)^h$ is:

Find $(\boldsymbol{v}, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2) \in \mathbb{R}^n \times \boldsymbol{\Lambda}_1 \times \boldsymbol{\Lambda}_2$ such that

$$\boldsymbol{K}\boldsymbol{u} = \boldsymbol{f} - \boldsymbol{N}^T\boldsymbol{\lambda}_1 - \boldsymbol{T}^T\boldsymbol{\lambda}_2 \tag{2.1}$$

$$(\boldsymbol{N}\boldsymbol{u} - \boldsymbol{d})^T(\boldsymbol{\lambda}_1 - \boldsymbol{\mu}_1) + \boldsymbol{u}^T\boldsymbol{T}^T(\boldsymbol{\lambda}_2 - \boldsymbol{\mu}_2) \geq 0,\ (\boldsymbol{\mu}_1, \boldsymbol{\mu}_2) \in \boldsymbol{\Lambda}_1 \times \boldsymbol{\Lambda}_2. \tag{2.2}$$

After computing $\mathbf{u}$ from (2.1) and substituting u into (2.2), we obtain the *algebraic dual formulation*:

$$\min\left\{\frac{1}{2}\boldsymbol{\lambda}^T\boldsymbol{A}\boldsymbol{\lambda} - \boldsymbol{\lambda}^T\boldsymbol{B}\right\}\ s.t.\ \boldsymbol{\lambda}_1 \geq 0,\ \ |\boldsymbol{\lambda}_1| \leq \boldsymbol{g},\ \boldsymbol{\lambda} = (\boldsymbol{\lambda}_1^T, \boldsymbol{\lambda}_2^T)^T, \tag{2.3}$$

where

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{N}\boldsymbol{K}^{-1}\boldsymbol{N}^T & \boldsymbol{N}\boldsymbol{K}^{-1}\boldsymbol{T}^T \\ \boldsymbol{T}\boldsymbol{K}^{-1}\boldsymbol{N}^T & \boldsymbol{T}\boldsymbol{K}^{-1}\boldsymbol{T}^T \end{pmatrix} \text{ and } \boldsymbol{B} = \begin{pmatrix} \boldsymbol{N}\boldsymbol{K}^{-1}\boldsymbol{f} - \boldsymbol{d} \\ \boldsymbol{T}\boldsymbol{K}^{-1}\boldsymbol{f} \end{pmatrix}.$$

The problem (2.3) is a *quadratic programming* problem that can be solved by several efficient algorithms.

3. Gauss-Seidel algorithm for solving the algebraic dual formulation

It is known that the matrix $\mathbf{A}$ is ill conditioned, and its diagonal blocks corresponding to the normal and tangential stress are closely related the dual Schur complement whose spectrum is not so ill conditioned.

The performance of duality algorithms may be improved if we split the normal and tangential stress. To exploit this fact, let us introduce a new notation for the natural block structure of the dual Hessian $\mathbf{A}$ and for the matrix $\mathbf{B}$, corresponding to normal stress, $\boldsymbol{\lambda}_N$, and tangential stress $\boldsymbol{\lambda}_T$:

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} \end{pmatrix},\ \ \boldsymbol{B} = \begin{pmatrix} \boldsymbol{B}_1 \\ \boldsymbol{B}_2 \end{pmatrix},\ \ \boldsymbol{\lambda} = \begin{pmatrix} \boldsymbol{\lambda}_1 \\ \boldsymbol{\lambda}_2 \end{pmatrix}.$$

The Gauss-Seidel algorithm for problem (2.3), constructs a sequence of approximations of $\boldsymbol{\lambda}_{\boldsymbol{N}}^{(i)}$ and $\boldsymbol{\lambda}_{\boldsymbol{T}}^{(0)}$ as follows:

Initialize $\boldsymbol{\lambda}_{\boldsymbol{N}}^{(0)} := \boldsymbol{g}^{(0)}$; $\boldsymbol{\lambda}_{\boldsymbol{T}}^{(0)} := 0$; $\boldsymbol{i} : 0$;

repeat

$\boldsymbol{i} := \boldsymbol{i} + 1$;

$\lambda_{\boldsymbol{T}}^{(\boldsymbol{i})} := (\boldsymbol{D} - \boldsymbol{L})^{-1} \cdot (\boldsymbol{U} \cdot \lambda_{\boldsymbol{T}}^{(\boldsymbol{i}-1)} + \boldsymbol{B}_2)$ such that $|\lambda_{\boldsymbol{T}}| \leq \phi \cdot \lambda_{\boldsymbol{N}}^{(\boldsymbol{i}-1)}$;

$\lambda_{\boldsymbol{N}}^{(\boldsymbol{i})} := \boldsymbol{A}_{11}^{-1} \cdot (\boldsymbol{B}_1 - \boldsymbol{A}_{12} \cdot \lambda_{\boldsymbol{T}}^{(\boldsymbol{i})})$ such that $\lambda_{\boldsymbol{N}} \geq 0$

until $|\boldsymbol{\lambda}^{(i)} - \boldsymbol{\lambda}^{(i-1)}| \leq Tol$;

where *Tol* is the chosen tolerance, the matrices $\boldsymbol{D}$, $-\boldsymbol{L}$ and $-\boldsymbol{U}$ represent the diagonal, strictly lower triangular and strictly upper triangular parts of $\boldsymbol{S}$, respectively, with $\boldsymbol{S} = \boldsymbol{A}_{21}\boldsymbol{A}_{11}^{-1}\boldsymbol{A}_{12} - \boldsymbol{A}_{22}$ is the Schur complement of the matrix $\boldsymbol{A}$.

Conclusions. The novelty of our approach in the present paper consists in the splitting of the normal stress and tangential stress, which leads to a better convergence of the solution, due to a better conditioned stiffness matrix. This batter conditioned matrix is based on the fact that these blocks diagonal matrices obtained, contain coefficients of the same size order.

4. Numerical examples. Contact of a long bar on a plane surface

This example has been considered by Raous [5] and it has the advantage of being very elementary and that of giving different contact states for given loading and coefficient of friction.

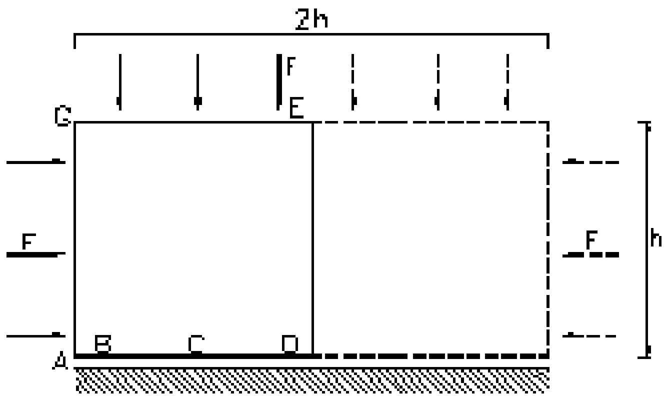

Fig.4.1 The geometry (h= 40 mm) and the loading

Table 1. Contact states for different loading cases

μ	F daN/mm^2	f daN/mm^2	Separate part AB mm	Sliding part BC mm	Stick part CD mm
1	10	-5	3.75	20	16.25
1	15	-5	5	20.75	7.5
0.2	10	-5	0	40	0
0.2	10	-15	0	22.5	17.5
0.2	10	-25	0	5	35

References

[1] Klarbring A., Mikeli A., Shillor M., *Frictional Contact Problem with Normal Compliance*, Int. J. Eng. Sci., **26** (1988), 811-832

[2] Ju J.W., Taylor R.L., *A Perturbed Lagrangian Formulation for the Finite Element Solution of Nonlinear Frictional Contact Problem*, Journal de Mécanique Theoretique et Appliquée, Spec. Issue, suppl. to vol. **7** (1998), 1-14

[3] Cocu M., *Existence of solutions of Signorini's problem with friction*, Int. J. Eng. Sci. 22, 5, 1984, 567-575

[4] Wriggers P., Simo J.C., *A note on tangent stiffness for fully nonlinear contact problems*, Comm. in App. Num. Math., **1** (1985), 199-203

[5] Raous M., Chabrand P., Lebon F., *Numerical methods for frictional contact problems and applications*, Journal de Mécanique Theoretique et Appliquée, Spec. issue, suppl. to vol. **7** (1998), 111-128

[6] Pop, N., *A Finite Element Solution for a Three-dimensional Quasistatic Frictional Contact Problem*, Rev. Roumaine des Sciences Tech. Serie Mec. Appliq. Editions de l'Academie Roumanie, Tom 42, 1-2, (1997), 209-218

North University of Baia Mare, Faculty of Science,
Department of Mathematic and Computer Science
Victoriei, 76, 430121 Baia Mare, Romania
E-mail address: nicpop@ubm.ro
E-mail address: izelina@myx.ro