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Order relation on the solutions sets of Z-conditional Cauchy equations on groups

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ABSTRACT. For a fixed pair of groups $(G, \circ)$ and $(H, *)$ and for all sets $Z \subset G \times G$ we consider the Z-conditional Cauchy equations

$$C_Z : f : G \longrightarrow H,\ f(x \circ y) = f(x) * f(y),\ (x, y) \in Z.$$

We prove that the family of the sets of solutions $\{S(C_Z)|Z \subset G \times G\}$ is a closure-system. This system is not a sublattice of $(\mathcal{P}(H^G), \subset)$ and generally it is not algebraic closure-system.

1. INTRODUCTION

Let $(G, \circ)$ and $(H, *)$ be two groups. The functions $f : G \to H$ which verifies the relations

(1.1) $\quad f(x \circ y) = f(x) * f(y)$

for all pairs $(x, y) \in G \times G$ are group morphisms, $f \in Hom(G, H)$. The equation (1.1) is the functional equation of morphisms or Cauchy equation.

If we look for the functions f which verifies (1.1) only for a subset of points $(x, y) \in G \times G$ we obtain other functional equations, which were called "conditional Cauchy equations" (J. Dhombres [1]) or Cauchy's functional equations on restricted domain" (M. Kuczma [5]).

2. MAIN RESULTS

In the paper we shall consider the family of Cauchy conditional equations for a pair of fixed groups (G, H) and we shall study the order structure of the set of the solutions, as a subset of the power set of H^G.

Definition 2.1. If $Z \subset G \times G$ is a fixed set then the functional equation:

$$(C_Z) : \begin{cases} f : G \to H \\ f(x \circ y) = f(x) * f(y),\ (x, y) \in Z \end{cases}$$

is called Z-conditional Cauchy equation or Cauchy equation conditioned by the set Z.

Remark 2.1. If we denote by $S(C_Z)$ the set of the solutions of the equation (C_Z), then for $Z = \emptyset$ we have $S(C_\emptyset) = H^G$ and if $Z = G \times G$ then $S(C_{G\times G}) = Hom(G, H)$.

Let $\{Z_i|\ i \in I\}$ be a family of subsets $Z_i \subset G \times G$ and $S(C_{Z_i})$ the set of the solutions of the equations (C_{Z_i}), $i \in I$.

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Proposition 2.1. *For any family $Z_i \subset G \times G$, $i \in I$ the following statements holds:*

a) If $Z_1 \subset Z_2$ then $S(C_{Z_2}) \subset S(C_{Z_1})$;

b) $\bigcup_{i \in I} S(C_{Z_i}) \subset S(C_Z)$;

c) $\bigcap_{i \in I} S(C_{Z_i}) = S(C_U)$ where $Z = \bigcap_{i \in I} Z_i$ and $U = \bigcup_{i \in I} Z_i$.

Proof. a) If $f \in S(C_{Z_2})$ then $f(x \circ y) = f(x) * f(y)$ for every $(x, y) \in Z_2 \subset Z_1$, it results $f \in S(C_{Z_1})$.

b) $Z = \bigcup_{i \in I} Z_i \subset Z_i \overset{a)}{\Rightarrow} S(C_{Z_i}) \subset S(C_Z)$, $i \in I \Rightarrow \bigcup_{i \in I} S(C_{Z_i}) \subset S(C_Z)$.

c) $Z_i \subset \bigcap_{i \in I} Z_i = U \overset{a)}{\Rightarrow} S(C_U) \subset S(C_{Z_i})$, $i \in I \Rightarrow S(C_U) \subset \bigcap_{i \in I} S(C_{Z_i})$.

If $f \in \bigcap_{i \in I} S(C_{Z_i})$ then $f(x \circ y) = f(x) * f(y)$, $(x, y) \in Z_i$, $i \in I \Rightarrow f(x \circ y) = f(x) * f(y)$; $(x, y) \in \bigcap_{i \in I} Z_i = U \Rightarrow f \in S(C_U)$. □

Remark 2.2. Generally the inclusion from b) is strict, which can be viewed from the following example:

$$(G, \circ) = (H, *) = (\mathbb{R}, +); \quad Z_1 = \mathbb{R} \times [0, \infty), \quad Z_2 = \mathbb{R} \times (-\infty, 0]$$
$$S(C_{Z_1}) = S(C_{Z_2}) = \{f : \mathbb{R} \to \mathbb{R} |\ f(x + y) = f(x) + f(y)\}$$
$$S(C_{Z_1 \cap Z_2}) = \{f : \mathbb{R} \to \mathbb{R} |\ f(0) = 0\}$$

Let $\mathcal{Z} = \{S(C_Z) |\ Z \subset G \times G\} \subset \mathcal{P}(H^G)$ be the family formed by those subsets of H^G which are solutions of some Z-conditional Cauchy equation.

Theorem 2.1. *The set Z form a closure-system on H^G of with the closure operator is $J : \mathcal{P}(H^G) \to \mathcal{P}(H^G)$ defined by $J(\mathcal{F}) = S(C_{U_\mathcal{F}})$, where $\mathcal{F} \subset \mathcal{P}(H^G)$ an arbitrary family and the set $U_\mathcal{F}$ is*

$$U_\mathcal{F} = \{(x, y) \in G \times G |\ f(x \circ y) = f(x) * f(y),\ f \in \mathcal{F}\}.$$

Proof. From Proposition 2.1 it follows that the intersection of any family in $\mathcal{Z}$ is in $\mathcal{Z}$, thus $\mathcal{Z}$ is a closure system.

If $\mathcal{F} \subset H^G$ then the closure operator is defined by:

$$J(\mathcal{F}) = \bigcap \{S(C_Z) |\ \mathcal{F} \subset S(C_Z)\}.$$

We shall assign for each function $f \in H^G$ the set U_f of the pairs on which f verifies the relations (1.1).

Let $U_f = \{(x, y) \in G \times G |\ f(x \circ y) = f(x) * f(y)\}$.

We have $f \in S(C_Z) \Leftrightarrow Z \subset U_f$, hence

$$\mathcal{F} \subset S(C_Z) \Leftrightarrow Z \subset U_f; \quad f \in \mathcal{F} \Leftrightarrow Z \subset \bigcap_{f \in \mathcal{F}} U_f = U_\mathcal{F}.$$

Using the definition of $J(\mathcal{F})$ and the relation c), from Proposition 2.1 we have:

$$J(\mathcal{F}) = \bigcap \{S(C_Z) |\ Z \subset U_\mathcal{F}\} = S(C_U)$$

where

$$U = \bigcup \{Z|\ Z \subset U_{\mathcal{F}}\} = U_{\mathcal{F}}.$$

Therefore:

$$U_{\mathcal{F}} = \bigcap_{f\in\mathcal{F}} U_f = \bigcap_{f\in\mathcal{F}} \{(x,y)|\ f(x\circ u) = f(x) * f(y)\} =$$
$$= \{(x,y) \in G\times G|\ f(x\circ y) = f(x)*f(y),\ f\in\mathcal{F}\}$$

and the proof is complete. □

Remark 2.3. Generally the closure system $\mathcal{Z}$ is a complete lattice, but it is not a sublattice of the lattice $(\mathcal{P}(H^G), \subset)$ because it is not closed to the union. For justification we give the following example:

Example 2.1. Let $(G,\circ) = (H,*) = (\mathbb{Z}_3, +)$ be the group of congruence classes modulo 3 and the sets: $Z_1 = \{(\widehat{0},\widehat{0})\}$, $Z_2 = \{(\widehat{1},\widehat{1})\}$.

We shall show that $S(C_{Z_1}) \cup S(C_{Z_2}) \notin \mathcal{Z}$.

We have

$$(C_{Z_2}) = \{f : \mathbb{Z}_3 \to \mathbb{Z}_3|\ f(\widehat{0}) = \widehat{0}\}$$
$$S(C_{Z_2}) = \{f : \mathbb{Z}_3 \to \mathbb{Z}_3|\ f(\widehat{0}) = 2f(\widehat{1})\}$$

each of these sets of solutions has 9 elements, hence in the union we have at most 18 elements (it can be established that there are 15 elements).

Because in $H^G = S(C_\emptyset)$ we have $3^3 = 27$ elements, obviously

$$S(C_{Z_1}) \cup S(C_{Z_2}) \neq S(C_\emptyset).$$

Now we suppose by contradiction that there exists $Z \subset G\times G$, $Z \neq \emptyset$ such that $S(C_{Z_1}) \cup S(C_{Z_2}) = S(C_Z)$. If $(x_0,y_0) \in Z$, then for every $f \in S(C_{Z_1}) \cup S(C_{Z_2})$ we have $f(x_0+y_0) = f(x_0) + f(y_0)$.

Take $f_1 \in S(C_{Z_1})$ defined thus: $f_1(\widehat{0}) = \widehat{0}$, $f_1(\widehat{1}) = \widehat{0}$, $f_1(\widehat{2}) = \widehat{1}$ and $f_2 \in S(C_{Z_2})$ defined by $f_2(\widehat{0}) = \widehat{1}$, $f_2(\widehat{1}) = \widehat{0}$, $f_2(\widehat{2}) = \widehat{0}$.

If $(x,y) \in \{(\widehat{1},\widehat{1}), (\widehat{1},\widehat{2}), (\widehat{2},\widehat{2})\} = A$ then

$$f_1(x+y) \neq f_1(x) + f_1(y).$$

If $(x,y) \in \{(\widehat{0},\widehat{0}), (\widehat{0},\widehat{1}), (\widehat{1},\widehat{2}), (\widehat{2},\widehat{0})\} = B$ then

$$f_2(x+y) \neq f_2(x) + f_2(y).$$

Hence $(x_0,y_0) \notin A$ and $(x_0,y_0) \notin B$, but $A\cup B = \mathbb{Z}_3\times\mathbb{Z}_3$ it results $(x_0,y_0) \notin \mathbb{Z}_3\times\mathbb{Z}_3$, which is contradiction with our hypothesis $Z \neq \emptyset$.

Remark 2.4. Generally the closure-system $\mathcal{Z}$ from Theorem 2.1, is not an algebraic closure-system. We will justify this statement with an example.

Example 2.2. Consider Cauchy's equations on the group $(\mathbb{Z}, +)$ of integers numbers, more precisely $(G,\circ) = (H,*) = (\mathbb{Z},+)$ and the family of the solutions of Cauchy's Z-equation.

$$\mathcal{Z} = \{S(C_Z)|\ Z \subset \mathbb{Z}\times\mathbb{Z}\}.$$

We show that the closure system $\mathcal{Z}$ is not algebraic. For this we shall construct a directed family to the right $\mathcal{D} \subset \mathcal{Z}$, for which $\bigcup_{D\in\mathcal{D}} D \notin \mathcal{Z}$.

Let $Z_n = \mathbb{Z}\times n\cdot\mathbb{Z} \subset \mathbb{Z}\times\mathbb{Z}$; $n \in \mathbb{N}^*$ and the family $\mathcal{D} = \{S(C_{Z_n})|\ n \in \mathbb{N}^*\}$. $\mathcal{D}$ is a directed family (for every $m, n \in \mathbb{N}^*$ there exists $k = m \cdot n$ such that $Z_k \subset Z_m$ and $Z_k \subset Z_n$ hence there exists k such that $S(C_{Z_m}) \subset S(C_{Z_k})$ and $S(C_{Z_n}) \subset S(C_{Z_k})$.

Suppose by contradiction that there exists $Z \subset \mathbb{Z} \times \mathbb{Z}$ such that

$$\bigcup_{n\in\mathbb{N}^*} S(C_{Z_n}) = S(C_Z).$$

If $Z \neq \emptyset$ and $(x_0, y_0) \in Z$, then

$$f(x_0 + y_0) = f(x_0) + f(y_0), \quad f \in \bigcup_{n\in\mathbb{N}^*} S(C_{Z_n}).$$

For $n \geq 3$ we define the functions $f_n : \mathbb{Z} \to \mathbb{Z}$,

$$f_n(x) = \begin{cases} k & \text{if} \quad x = n \cdot k \\ k + n - 1 & \text{if} \quad x = n \cdot k + r \end{cases}$$

where $k \in \mathbb{Z}$, $0 < r < n$. First we prove that $f_n \in S(C_{Z_n})$.
If $(x, n \cdot y) \in Z_n$ then

$$f_n(x + n \cdot y) = \begin{cases} x_1 + y, & \text{if } x = n \cdot x_1,\ x_1 \in \mathbb{Z} \\ x_1 + y + n - 1, & \text{if } x = n \cdot x_1 + r,\ x_1 \in \mathbb{Z},\ r \in \{1, \dots, n-1\} \end{cases}$$

$$f_n(x) = \begin{cases} x_1, & \text{if } x = n \cdot x_1,\ x_1 \in \mathbb{Z} \\ x_1 + n - 1, & \text{if } x = n \cdot x_1 + r,\ x_1 \in \mathbb{Z},\ r \in \{1, \dots, n-1\} \\ & f_n(n \cdot y) = y \end{cases}$$

thus $f_n(x + n \cdot y) = f_n(x) + f_n(n \cdot y)$, for all $x, y \in \mathbb{Z}$.

From $(x_0, y_0) \in \mathbb{Z}$ we have $f_n(x_0+y_0) = f_n(x_0)+f_n(y_0)$ and we get that x_0 or y_0 is divisible by n: If $x_0 = n\cdot k_1+r_1,\ y_0 = n\cdot k_2+r_2,\ k_1, k_2 \in \mathbb{Z}, r_1, r_2 \in \{1, \dots, n-1\}$ then $x_0 + y_0 = n \cdot (k_1 + k_2) + r_1 + r_2,\ r_1 + r_2 \in \{1, \dots, n, \dots, 2n - 2\}$, so $f(x_0) = k_1 + n - 1,\ f(y_0) = k_2 + n - 1$ and

$$f(x_0 + y_0) \in \{k_1 + k_2 + 1,\ k_1 + k_2 + n - 1,\ k_1 + k_2 + 1 + n - 1\},$$

hence $f(x_0) + f(y_0) = k_1 + k_2 + 2n - 2 \neq f(x_0 + y_0)$.
Hence at least one of these numbers is divisible by an infinity of numbers, thus it is zero.

We showed that $Z \subset \{(x, 0), (0, x)|\ x \in \mathbb{Z}\} = U$ hence

$$S(C_U) \subset S(C_Z) = \bigcup_{n\in\mathbb{N}^*} S(C_{Z_n}).$$

But the function $f_0 : \mathbb{Z} \to \mathbb{Z}$

$$f_0(x) = \begin{cases} 0, & x = 0 \\ 1, & x \neq 0 \end{cases}$$

is from $S(C_U)$ but it is not in $S(C_{Z_n})$ for every $n \in \mathbb{N}^*$ $(1 = f_0(1 + n) \neq f_0(1) + f_0(n) = 2)$.

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