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Fixed points, upper and lower fixed points: abstract Gronwall lemmas

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ABSTRACT. The aim of this paper is to study four basic problems of operatorial inequalities.

1. INTRODUCTION

Let $(X, \leq)$ be an ordered set and $A : X \to X$ an operator. We denote
$F_A := \{x \in X|\ A(x) = x\}$ - the fixed point set of A;
$(UF)_A := \{x \in X|\ A(x) \leq x\}$ - the upper fixed point set of A;
$(LF)_A := \{x \in X|\ A(x) \geq x\}$ - the lower fixed point set of A.
In this paper we shall study the following problems:
Problem 1. If $F_A = \{x_A^*\}$, in which conditions we have that

$$(LF)_A \leq x_A^* \leq (UF)_A \ ?$$

Problem 2. If $F_A \neq \emptyset$, in which conditions there exists $\varphi : X \to F_A$ such that
(i) $x \leq A(x) \Rightarrow x \leq \varphi(x)$?
(ii) $x \geq A(x) \Rightarrow x \geq \varphi(x)$?
Problem 3. If $F_A = \{x_A^*\}$ and $B : X \to X$ is an operator such that $F_B = \{x_B^*\}$ and $A \leq B$ in which conditions we have that

$$x \leq A(x) \Rightarrow x \leq x_B^* \ ?$$

Problem 4. If $F_A \neq \emptyset$, $F_B \neq \emptyset$, $A \leq B$, in which conditions there exists $\psi : X \to F_B$ such that

$$x \leq A(x) \Rightarrow x \leq \psi(x) \ ?$$

Throughout this paper we follow the terminology and notations in I. A. Rus [40]. For the convenience of the reader we shall recall some of them.

2. PICARD AND WEAKLY PICARD OPERATORS

Let $(X, \to)$ be an L-space and $A : X \to X$ an operator.

Definition 2.1. (I. A. Rus [40]). The operator A is Picard operator (PO) if
(i) $F_A = \{x_A^*\}$;
(ii) $A^n(x) \to x_A^*$ as $n \to \infty$, $\forall\ x \in X$.

Definition 2.2. (I. A. Rus [40]). The operator A is weakly Picard operator (WPO) if the sequence $(A^n(x))_{n\in\mathbb{N}}$ converges for all $x \in X$ and the limit (which may depend on x) is a fixed point of A.

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Definition 2.3. (I. A. Rus [40]). If $A : X \to X$ is an WPO, then we consider the operator $A^\infty : X \to X$ defined by $A^\infty(x) = \lim_{n\to\infty} A^n(x)$.

Remark 2.1. It is clear that $A^\infty(X) = F_A$.

Remark 2.2. For some examples of POs and WPOs in a variety of L-spaces see [40] pp. 194-195, 198-201, Sz. András [1], A. Buică [6], [7], C. Crăciun [10], V. Dincuţă [12], N. Lungu [25], L. Lungu and I. A. Rus [26], [27], V. Mureşan [29], [30], I. A. Rus [38], [39],...

Remark 2.3. For some examples of L-spaces see I. A. Rus [40], P. P. Zabreijko [48], S. Keikkilä and S. Seikkala [17], M. Kwapisz [21], J. Schröder [42].

3. Problem 1

Let $(X, \to)$ be an L-space and, $\leq$ an ordered relation on X. If the following implication holds

$$x_n \leq y_n,\ x_n \to x^*,\ y_n \to y^* \text{ as } n \to \infty \Rightarrow x^* \leq y^*, \tag{3.1}$$

then, by definition, $(X, \to, \leq)$ is an ordered L-space.

The following result is given in [40].

Lemma 3.1. (Abstract Gronwall lemma). *Let $(X, \to, \leq)$ be an ordered L-space and $A : X \to X$ an operator. We suppose that*

(i) A is PO;

(ii) A is increasing.

Then

$$(LF)_A \leq x_A^* \leq (UF)_A,$$

where x_A^* is the unique fixed point of A.

Proof. Let $x \in (LF)_A$. From (ii) we have that

$$x \leq A(x) \leq \cdots \leq A^n(x),\ \forall\ n \in \mathbb{N}^*.$$

From (i), $A^n(x) \to x_A^*$ as $n \to \infty$. Since the ordered relation is closed, by (3.1) we obtain $x \leq x_A^*$. In a similar way we prove that

$$x \in (LF)_A \Rightarrow x \geq x_A^*.$$

□

Remark 3.1. Condition (ii) implies that $(LF)_A$ and $(UF)_A$ are invariant subsets for A. So, in Lemma 3.1, instead of condition (i) we can put the condition

(i') the restriction of A to $(UF)_A \cup (LF)_A$ is a PO.

From Lemma 3.1 we have

Theorem 3.1. *Let X be a sequentially complete, Hausdorff separated, ordered uniform space and $A : X \to X$ an increasing operator. Let $(d_\alpha)_{\alpha\in\mathcal{A}}$ be a separated saturated set of pseudometrics defining the uniform structure on X. We suppose that there exist an operator $\varphi : \mathcal{A} \to \mathcal{A}$ and the positive numbers $\lambda_\alpha > 0$ such that*

$$d_\alpha(A(x), A(y)) \leq \lambda_\alpha d_{\varphi(\alpha)}(x, y),\ \textit{for all } \alpha \in \mathcal{A} \textit{ and } x, y \in X,$$

and the series

$$d_\alpha(x,y)+\sum_{n=1}^{\infty}\lambda_\alpha\lambda_{\varphi(\alpha)}\dots\lambda_{\varphi^{n-1}(\alpha)}d_{\varphi^n(\alpha)}(x,y)$$

are convergent for each $\alpha\in\mathcal{A}$ and all $x,y\in X$.

Then the operator A is PO and we have

$$(LF)_A\le x_A^*\le(UF)_A.$$

Proof. Consider the ordered L-space $(X,\xrightarrow{(d_\alpha)_{\alpha\in\mathcal{A}}},\le)$. By a theorem of N. Gheorghiu (see [41], pp. 28) the operator A is PO. The proof follows by Lemma 3.1. □

Theorem 3.2. *Let $(X,d,\le)$ be an ordered complete metric space and $A:X\to X$ an increasing operator. We suppose that there exist $a_i>0$, $i=1,2,3$, with $a_1+a_2+a_3<1$, such that*

$$d(A(x),A(y))\le a_1d(x,y)+a_2d(x,A(x))+a_3d(y,A(y)),\ \forall\ x,y\in X.$$

Then A is PO and

$$(LF)_A\le x_A^*\le(UF)_A.$$

Proof. Consider the ordered L-space $(X,\xrightarrow{d},\le)$. By a theorem of Ćirić-Reich-Rus (see I. A. Rus [38], pp. 10, 48-50) the operator A is PO. Now we apply Lemma 3.1. □

Theorem 3.3. (V. Lakshmikantham, S. Leela and A. A. Martynyuk (1989; [23])). *Let $(X,d,\le)$ be an ordered complete metric space and $A:X\to X$ an increasing operator. We suppose that there exists $n_0\in\mathbb{N}^*$ such that A^{n_0} is a contraction. Then*

$$(LF)_A\le x_A^*\le(UF)_A.$$

Proof. By Theorem 1.3.2 in [38] the operator A is PO in $(X,\xrightarrow{d})$. The proof follows by Lemma 3.1. □

Theorem 3.4. (K. Valeev (1973; [45]), L. Losonczi (1973; [24]), V. Ya. Stetsenko and M. Shaaban (1986; [43])). *Let $(X,\|\cdot\|,\le)$ be an ordered Banach space and $A:X\to X$ an increasing PO. Then*

$$(LF)_A\le x_A^*\le(UF)_A.$$

Remark 3.2. In Lemma 3.1, instead of $(X,\to,\le)$ we can put one of the following:

- $(X,d,\le)$ an ordered complete metric space;
- $(X,d,\le)$ an ordered complete K-metric space;
- $(X,d,\le)$ an ordered complete 2-metric space;
- $(X,F,\tau,\le)$ an ordered complete probabilistic metric space.

Instead of condition (i) we can put any condition in the above spaces which implies that A is PO. For some examples, see [40].

Remark 3.3. Let $(X,\le)$ be an ordered set and $A:X\to X$ an operator. In order to have a concrete Gronwall lemma we follow the following algorithm:

- we examine if A is increasing

- we choose an L-space structure on X which satisfies (3.1)
- we examine if A is PO on $(X, \to)$
- we "determine" the unique fixed point of A, x_A^*.

The last step in the above algorithm is a difficult problem in the way to obtain a concrete Gronwall lemma (see N. Lungu and I. A. Rus [27]). As an example we have

Theorem 3.5. (I. A. Rus [40]). *Let $(X, +, \leq, \to)$ be an ordered L-space group. Let $A : X \to X$ an operator and $y \in X$. We suppose that*

(*i*) *A is PO;*

(*ii*) *A is additive, continuous and increasing;*

(*iii*) *the Neumann series $\sum_{k\in\mathbb{N}} A^k(y)$ converges.*

Then we have

a) $x \leq A(x) + y \Rightarrow x \leq \sum_{k\in\mathbb{N}} A^k(y)$;

b) $x \geq A(x) + y \Rightarrow x \geq \sum_{k\in\mathbb{N}} A^k(y)$.

Remark 3.4. For other abstract and concrete Gronwall lemmas see [1], [3], [5], [9], [13], [14], [15], [21], [22], [23], [28], [31], [46], [47], [50].

Remark 3.5. For Gronwall lemmas via Picard and weakly Picard operators see Sz. András [1], A. Buică [6], [7], C. Crăciun [10], V. Dincuţă [12], N. Lungu [25], N. Lungu and I. A. Rus [26], [27], V. Mureşan [29], [30], I. A. Rus [39], [40].

4. Problem 2

We begin with

Lemma 4.1. (I. A. Rus [40]). *Let $(X, \to, \leq)$ be an ordered L-space and $A : X \to X$ an operator. We suppose that*

(*i*) *A is WPO;*

(*ii*) *A is increasing.*

Then

a) $x \leq A(x) \Rightarrow x \leq A^\infty(x)$;

b) $x \geq A(x) \Rightarrow x \geq A^\infty(x)$.

Proof. (a). Let $x \leq A(x)$. From (ii) we have that

$$x \leq A^n(x),\ \forall\, n \in \mathbb{N}.$$

Since the ordered relation is closed, we obtain $x \leq A^\infty(x)$.

(b). In a similar way we can prove (b). □

Remark 4.1. *$A^\infty(x)$ is the minimum element of $F_A \cap [x, \cdot)$, where $[x, \cdot) := \{y \in X \mid y \geq x\}$.*

From Lemma 4.1 we have

Lemma 4.2. (monotone iteration lemma). *Let $(X, \to, \leq)$ be an ordered L-space and $A : X \to X$ an operator. We suppose that:*

(*i*) *A is WPO;*

(ii) *A is increasing;*

(iii) *there exist $x \in (LF)_A$, $y \in (UF)_A$ such that $x \leq y$.*

Then

(a) $x \leq A(x) \leq \cdots \leq A^n(x) \leq \cdots \leq A^\infty(x) \leq A^\infty(y) \leq \cdots \leq A^n(y) \leq \cdots \leq A(y) \leq y$.

(b) $A^\infty(x)$ is the minimum element of $F_A \cap [x,y]$ and $A^\infty(y)$ is the maximum element of $F_A \cap [x,y]$.

Remark 4.2. Instead of (i) we can put the following condition

(i) the restriction of A to $(LF)_A \cup (UF)_A$ is WPO.

Theorem 4.1. *Let $(X, \leq)$ be an ordered set and $A : X \to X$ an operator. We suppose that*

(i) *A is increasing;*

(ii) *$x_n \in X$, $x_n \leq x_{n+1}$, $n \in \mathbb{N} \Rightarrow \exists \sup\limits_{n\in\mathbb{N}} x_n$ and*

$$A\left(\sup_{n\in\mathbb{N}} x_n\right) = \sup_{n\in\mathbb{N}} A(x_n).$$

Then

$$x \leq A(x) \Rightarrow x \leq \sup_{n\in\mathbb{N}} A^n(x).$$

Proof. Let $C(X) := \{(x_n)_{n\in\mathbb{N}} |\ x_n \leq x_{n+1},\ n \in \mathbb{N}\}$, and $\mathrm{Lim}(x_n)_{n\in\mathbb{N}} := \sup\limits_{n\in\mathbb{N}} x_n$. Then $(X, C(X), \mathrm{Lim})$ is an L-space and $A : (LF)_A \to (LF)_A$ is WPO. We remark that $A^\infty(x) = \sup\limits_{n\in\mathbb{N}} A^n(x)$, for $x \in (LF)_A$.

Now the proof follows Lemma 4.1 and Remark 4.2. □

Theorem 4.2. *Let $(X, d, \leq)$ be an ordered complete metric space, $A : X \to X$ and $\alpha \in]0,1[$. We suppose that*

(i') *$d(A^2(x), A(x)) \leq \alpha d(x, A(x)),\ \forall\ x \in X$;*

(i'') *A is closed;*

(ii) *A is increasing.*

Then A is WPO in $(X, \xrightarrow{d})$ and we have (a) and (b) in Lemma 4.1.

Proof. The conditions (i'), (i'') imply that A is WPO in $(X, \xrightarrow{d})$ (see [40]). The proof follows by Lemma 4.1. □

Theorem 4.3. *Let $(X, d, \leq)$ be an ordered complete metric space and $\theta : X \to \mathbb{R}_+$ a functional. We suppose that*

(i') *the operator A satisfies the Caristi condition relative to θ, i.e.,*

$$d(x, A(x)) \leq \theta(x) - \theta(A(x)),\ \forall\ x \in X;$$

(i'') *the operator A is closed;*

(ii) *A is increasing.*

Then A is WPO in $(X, \xrightarrow{d})$ and we have (a) and (b) in Lemma 4.1.

Proof. From (i') we have that

$$\sum_{n\in\mathbb{N}} d(A^n(x), A^{n+1}(x)) \leq \theta(x),\ \forall\ x \in X.$$

This implies that $(A^n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence, so $(A^n(x)))n \in \mathbb{N}$ converges for all $x \in X$. From (i"), the limit is the fixed point of A. We are in the conditions of Lemma 4.1. □

From Lemma 4.2 we have

Theorem 4.4. (S. Carl and S. Heikkilä [8]). *Let $[x, y]$ be a nonempty order interval in the ordered metric space $(X, d, \leq)$ and $A : [x, y] \to [x, y]$ be an increasing operator. If $(A(x_n))_{n\in\mathbb{N}}$ converges whenever $(x_n)_{n\in\mathbb{N}}$ is monotone sequence in $[x, y]$, then A has the least fixed point x_* and the greatest fixed point x^* in $[x, y]$.*

Theorem 4.5. (R. Precup [34]). *Let $(X, \|\cdot\|, \leq)$ be an ordered Banach space, $[x, y] \subset X$ an order interval $(x < y)$ and $A : [x, y] \to [x, y]$ an increasing operator. Assume one of the following conditions holds:*
(i) $K := \{x \in X|\ x \geq 0\}$ is a regular cone.
(ii) K is a normal cone and A is completely continuous.

Then there exist $x_*, x^* \in [x, y]$, $x_{\leq} x^*$, $x_*, x^* \in F_A$ and $A^n(x) \xrightarrow{\|\cdot\|} x_*$, $A^n(y) \xrightarrow{\|\cdot\|} x^*$.

Remark 4.3. For other results for Problem 2 see H. Amann [2], E. N. Dancer and P. Hess [11], S. Heikkilä and S. Seikkala [17], P. Hess [18], M. W. Hirsch [19], M. A. Krasnoselskii and P. Zabreiko [20], M. Kwapisz [21], J. Schröder [42].

5. Problem 3

For Problem 3 we have the following general results

Lemma 5.1. (Abstract Gronwall-comparison lemma). *Let $(X, \to, \leq)$ be an ordered L-space and $A, B : X \to X$ two operators. We suppose that*
(i) A and B are POs;
(ii) A is increasing;
(iii) $A \leq B$.
Then

$$x \leq A(x) \Rightarrow x \leq x^*_B.$$

Proof. Let $F_A = \{x^*_A\}$ and $F_B = \{x^*_B\}$. From Lemma 3.1, $x \leq A(x)$ implies that $x \leq x^*_A$.

Now we prove that $x^*_A \leq x^*_B$.

For this, by induction we prove that

(5.1) $\quad x^*_A \leq B^n(x^*_A),\ \forall\ n \in \mathbb{N}.$

For $n = 1$, we have $x^*_A = A(x^*_A) \leq B(x^*_A)$.

From $x^*_A \leq B^k(x^*_A)$, we have

$$x^*_A = A(x^*_A) \leq A(B^k(x^*_A)) \leq B(B^k(x^*_A)) = B^{k+1}(x^*_A).$$

So, we have (5.1).

Since B is PO we have that

$$B^n(x) \to x^*_B \text{ as } n \to \infty,\ \forall\ x \in X.$$

From (5.1) we have $x^*_A \leq x^*_B$. □

From the proof of Lemma 5.1 we have

Lemma 5.2. *Let $(X, \to, \leq)$ be an ordered L-space and $A, B : X \to X$ two operators. We suppose that*
(i) *A and B are POs;*
(ii) *A and B are increasing;*
(iii) $x = A(x) \Rightarrow x \leq B(x)$.
Then

$$x \leq A(x) \Rightarrow x \leq x^*_B.$$

Remark 5.1. If in Lemma 5.2, instead of condition (iii) we put
(iii') $x = A(x) \Rightarrow x \geq B(x)$,
then we have

$$x \leq A(x) \Rightarrow x \geq x^*_B.$$

Remark 5.2. See Remark 3.2.

For example from Lemma 5.2 we have

Theorem 5.1. *Let $(X, d, \leq)$, $d(x, y) \in \mathbb{R}^m$, be a complete generalized metric space and $A, B : X \to X$ two operators. We suppose that*
(i') *A and B are contractions;*
(ii) *A and B are increasing;*
(iii) $x = A(x) \Rightarrow x \leq B(x)$.
Then A and B are POs and

$$x \leq A(x) \Rightarrow x \leq x^*_B.$$

Proof. From (i) and Perov's theorem it follows that A and B are POs. □

Remark 5.3. For applications of Lemma 5.2 to Volterra integral equations see N. Lungu and I. A. Rus [27].

Remark 5.4. Let $(X, \leq)$ be an ordered L-space and $A : X \to X$ an operator.

In order to have a concrete Gronwall-comparison lemma we follow the following algorithm:

- We examine if A is increasing.
- We choose an L-space structure on X which satisfies (3.1).
- We choose an increasing operator B which satisfies condition (iii) in Lemma 5.2.
- We examine if A and B are POs on $(X, \to)$.
- We "determine" the unique fixed point of B, x^*_B.

For example we have

Theorem 5.2. *Let $(X, +, \leq, \to)$ be an ordered L-space group. Let A and B two operators and $y \in X$. We suppose that*
(i) *$A + y$ and B are PO;*
(ii) *A and B are increasing;*
(iii) *B is additive, continuous and the Neumann series $\sum_{k \in \mathbb{N}} A^k(y)$ converges;*

(*iv*) $x = A(x) + y \Rightarrow x \leq B(x) + y$.

Then

$$x \leq A(x) + y \Rightarrow x \leq \sum_{k \in \mathbb{N}} B^k(y).$$

Proof. (i) and (iii) imply that the operator $B + y$ is PO. Condition (ii) implies that $A + y$ and $B + y$ are increasing. So, we are in the conditions of Lemma 5.2 for the operators $A + y$ and $B + y$. On the other hand we remark that $\sum_{k \in \mathbb{N}} B^k(y)$ is the unique solution of the equation $x = B(x) + y$. □

6. Problem 4

We have

Lemma 6.1. Let $(X, \to, \leq)$ be an ordered L-space and $A, B : X \to X$ two operators. We suppose that

(*i*) A and B are WPOs;

(*ii*) A is increasing;

(*iii*) $A \leq B$.

Then

(*a*) $x \leq y \Rightarrow A^\infty(x) \leq B^\infty(y)$;

(*b*) if in addition B is increasing

then

$$x \leq A(x) \Rightarrow x \leq B^\infty(x).$$

Proof. (a) Let $x \leq y$. From (ii) we have $A(x) \leq A(y)$. From (iii) it follows that $A(x) \leq B(y)$. By induction we have that $A^n(x) \leq B^n(y)$, for all $n \in \mathbb{N}$. Since the ordered relation, $\leq$ is closed we have $A^\infty(x) \leq A^\infty(y)$.

(b). Let $x \in X$, such that $x \leq A(x)$. From Lemma 4.1 we have that $x \leq A^\infty(x)$. From (a) in the case $y = x$, it follows that $A^\infty(x) \leq B^\infty(x)$, so, $x \leq B^\infty(x)$. □

Remark 6.1. Comments and applications of Lemma 6.1 will be presented elsewhere.

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