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Cubature formulas of high degree of exactness

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ABSTRACT. In this paper we use the multivariate interpolation for construction of cubature formulas of high degree of exactness, when the domain of interpolation is a rectangle. These formulas are of Gauss-Christoffel-Stancu type. The coordinates of the Gaussian nodes are determined by using the roots of the orthogonal polynomials of Christoffel-Szegö.

1. INTRODUCTION

It is known that the interpolation of functions of several variables is defined by means of a limit process and since such a process cannot be carried out on the computer, we must replace them by finite processes.

Our basic problem is to calculate the definite multiple integral of a given function f of several variables.

We want to construct cubature formulas, which represent approximate procedures for computing the value of a double integral, by using a given set of numerical values of the integrand and its partial derivatives on some given points of a region Ω.

2. USE OF THE BIVARIATE INTERPOLATION FORMULA OF LAGRANGE-HERMITE

In order to construct cubature formulas of interpolatory type, we shall use the Lagrange-Hermite interpolation formula of several variables.

We consider that we have a rectangular grid of nodes determined by the intersection of distinct straight lines $x = x_i \in [a, a']$ $(i = \overline{1, m+1})$, $y = y_j \in [b, b']$ $(j = \overline{1, n+1})$. We consider also the points α_i and a_r from $[a, a']$, where $i = \overline{1, m+1}$, $r = \overline{1, n_1}$, as well as the following points from $[b, b'] : \beta_j (j = \overline{1, n+1})$ and $b_s (s = \overline{1, n_2})$.

Let us attach to a_r the multiplicity p_r and to b_s the multiplicity q_s.

Now we introduce the notations:

$$M = 2m + p_1 + \cdots + p_{n_1} + 1, \; N = 2n + q_1 + \cdots + q_{n_2} + 1,$$

$$u(x) = \prod_{i=1}^{m+1} (x - x_i), \; e(x) = \prod_{i=1}^{m+1} (x - \alpha_i), \; A(x) = \prod_{r=1}^{n_1} (x - a_r)^{p_r},$$

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$$v(y)=\prod_{j=1}^{n+1}(y-y_j),\ g(y)=\prod_{j=1}^{n+1}(y-\beta_j),\ B(y)=\prod_{s=1}^{n_2}(y-b_s)^{q_s},$$
$$u_i(x)=\frac{u(x)}{x-x_i},\ e_i(x)=\frac{e(x)}{x-\alpha_i},\ A_r(x)=\frac{A(x)}{(x-a_r)^{p_r}},$$
$$v_j(y)=\frac{v(y)}{y-y_j},\ g_j(y)=\frac{g(y)}{y-\beta_j},\ B_s(y)=\frac{B(y)}{(y-b_s)^{q_s}}.$$

The corresponding Lagrange-Hermite double interpolation formula is of the following form:

$$f(x,y)=(L_{M,N}f)(x,y)+(R_{M,N}f)(x,y), \tag{2.1}$$

where the first term from the second side represents the Lagrange-Hermite interpolation polynomial which has the expression (see [5]):

$$\begin{aligned}(L_{M,N}f)(x,y)&=\sum_{i=1}^{m+1}\sum_{j=1}^{n+1}l_i^{(1)}(x)g_j^{(1)}(y)f(x_i,y_j)+\\
&+\sum_{i=1}^{m+1}\sum_{j=1}^{n+1}l_i^{(1)}(x)g_j^{(2)}(y)f(x_i,\beta_j)+\sum_{i=1}^{m+1}\sum_{j=1}^{n+1}l_i^{(2)}(x)g_j^{(1)}(y)f(\alpha_i,y_j)+\\
&+\sum_{i=1}^{m+1}\sum_{j=1}^{n+1}l_i^{(2)}(x)g_j^{(2)}(y)f(\alpha_i,\beta_j)+\sum_{i=1}^{n_1}\sum_{\nu=0}^{p_i-1}\sum_{j=1}^{n+1}l_{i,\nu}(x)g_j^{(1)}(y)f^{(\nu,0)}(a_i,y_j)+\\
&+\sum_{i=1}^{m+1}\sum_{j=1}^{n+1}\sum_{\mu=0}^{q_j-1}l_i^{(2)}(x)h_{j,\mu}f^{(0,\mu)}(x_i,b_j)+\\
&+\sum_{i=1}^{m+1}\sum_{\nu=0}^{p_i-1}\sum_{j=1}^{n+1}l_{i,\nu}(x)g_j^{(2)}(y)f^{(\nu,0)}(a_i,\beta_j)+\\
&+\sum_{i=1}^{m+1}\sum_{j=1}^{n+1}\sum_{\mu=0}^{p_j-1}l_i^{(2)}(x)h_{j,\mu}(y)f^{(0,\mu)}(\alpha_i,b_j)+\\
&+\sum_{i=1}^{m+1}\sum_{\nu=0}^{p_i-1}\sum_{j=1}^{n+1}\sum_{\mu=0}^{q_j-1}l_{i,\nu}(x)h_{j,\mu}(y)f^{(\nu,\mu)}(a_i,b_j),\end{aligned}$$

where

$$l_i^{(1)}(x)=\frac{u_i(x)e(x)A(x)}{u_i(x_i)e(x_i)A(x_i)},\ l_i^{(2)}(x)=\frac{u(x)e_i(x)A(x)}{u(\alpha_i)e_i(\alpha_i)A(\alpha_i)},$$
$$l_{i,\nu}(x)=\sum_{t=0}^{p_i-\nu-1}\frac{(x-a_i)^\nu}{\nu!}\left[\frac{(x-a_i)^t}{t!}\left(\frac{1}{C_i(x)}\right)_{a_i}^{(t)}\right]C_i(x),$$
$$g_j^{(1)}(y)=\frac{v_j(y)g(y)B(y)}{v_j(y_j)g(y_j)B(y_j)},\ g_j^{(2)}(y)=\frac{v(y)g_j(y)B(y)}{v(\beta_j)g_j(\beta_j)B(\beta_j)},$$
$$h_{j,\mu}(y)=\sum_{\tau=0}^{q_j-\mu-1}\frac{(y-b_j)^\mu}{\mu!}\left[\frac{(y-b_j)^\tau}{\tau!}\cdot\left(\frac{1}{D_j(y)}\right)_{b_j}^{(\tau)}\right]D_j(y)$$

and $C_i(x) = u(x)e(x)A_i(x),\ D_j(y) = v(y)g(y)B_j(y)$.

The remainder of the interpolation formula (2.1) can be expressed, by means of three partial divided differences, under the form

$$\begin{aligned}(2.2)\qquad (R_{M,N}f)(x,y) &= u(x)e(x)A(x)\big[x, x_1, \ldots, x_{m+1}, \alpha_1, \ldots,\\ &\alpha_{m+1}, (a_1)^{p_1}, \ldots, (a_{n_1})^{p_{n_1}}; f(t,y)\big] + v(y)g(y)B(y)\big[y, y_1, \ldots, y_{n+1},\\ &\beta_1, \ldots, \beta_{n+1}, (b_1)^{q_1}, \ldots, (b_{n_2})^{q_{n_2}}, f(x,z)\big] - u(x)e(x)A(x)v(y)g(y)B(y)\cdot\\ &\cdot\left[\begin{matrix} x, x_1, \ldots, x_{m+1}, \alpha_1, \ldots, \alpha_{m+1}, (a_1)^{p_1}, \ldots, (a_{n_1})^{p_{n_1}} \\ y, y_1, \ldots, y_{n+1}, \beta_1, \ldots, \beta_{n+1}, (b_1)^{q_1}, \ldots, (b_{n_2})^{q_{n_2}} \end{matrix}; f(t,z)\right].\end{aligned}$$

The notation $(c)^r$ means that c is repeated r times.

If we suppose that the function $f(x,y)$ has continuous partial derivatives of the order $(M+1, N+1)$, then by applying the laws of the means for divided differences, we obtain the following expression for this remainder:

$$\begin{aligned}(R_{M,N}f)(x,y) = &\frac{u(x)e(x)A(x)}{(M+1)!} f^{(M+1,0)}(\xi, y) + \frac{v(y)g(y)B(y)}{(N+1)!} f^{(0,N+1)}(x,\eta) -\\ &- \frac{u(x)e(x)v(y)g(y)A(x)B(y)}{(M+1)!(N+1)!} f^{(M+1,N+1)}(\xi,\eta),\end{aligned}$$

where $\xi \in (a, a')$ and $\eta \in (b, b')$.

3. Construction of cubature formulas of high degree of exactness

By using the general bivariate interpolation formula (2.1), we can construct a cubature formula for approximating the value of the double integral

$$(3.3)\qquad I_2(f) = \iint_{D_2} \omega(x,y) f(x,y)\, dx dy$$

where $\omega(x,y)$ is a bivariate weight function, while $D_2 = [a, a'] \times [b, b']$.

Theorem 3.1. *The cubature formula is given by*

$$\begin{aligned}(3.4)\qquad I_2(f) = &\sum_{i=1}^{m+1}\sum_{j=1}^{n+1} A_{i,j} f(x_i, y_j) + \sum_{i=1}^{m+1}\sum_{j=1}^{n+1} B_{i,j} f(x_i, \beta_j) +\\ &+ \sum_{i=1}^{m+1}\sum_{j=1}^{n+1} C_{i,j} f(\alpha_i, y_j) + \sum_{i=1}^{m+1}\sum_{j=1}^{n+1} D_{i,j} f(\alpha_i, \beta_j) +\\ &+ \sum_{i=1}^{n_1}\sum_{\nu=0}^{p_i-1}\sum_{j=1}^{n+1} E_{i,\nu,j} f^{\nu,0}(a_i, y_j) + \sum_{i=1}^{m+1}\sum_{j=1}^{n_2}\sum_{\mu=0}^{q_j-1} F_{i,j,\mu} f^{(0,\mu)}(x_i, b_j) +\\ &+ \sum_{i=1}^{n_1}\sum_{\nu=0}^{p_i-1}\sum_{j=1}^{n+1} G_{i,\nu,j} f^{(\nu,0)}(a_i, \beta_j) + \sum_{i=1}^{m+1}\sum_{j=1}^{n_2}\sum_{\mu=0}^{q_j-1} H_{i,j,\mu} f^{(0,\mu)}(\alpha_i, b_j) +\\ &+ \sum_{i=1}^{n_1}\sum_{\nu=0}^{p_i-1}\sum_{j=1}^{n_2}\sum_{\mu=0}^{q_j-1} I_{i,\nu,j,\mu} f^{(\nu,\mu)}(a_i, b_j),\end{aligned}$$

where we have

$$
\begin{aligned}
&A_{i,j} = I_2\left(l_i^{(1)}(x)g_j^{(1)}(y)\right),\ B_{i,j} = I_2\left(l_i^{(1)}(x)g^{(2)}(y)\right),\\
&C_{i,j} = I_2\left(l_i^{(2)}(x)g_j^{(1)}(y)\right),\ D_{i,j} = I_2\left(l_i^{(2)}(x)g_j^{(2)}(y)\right),\\
&E_{i,\nu,j} = I_2\left(l_{i,\nu}(x)g_j^{(1)}(y)\right),\ F_{i,j,\mu} = I_2\left(l_i^{(2)}(x)h_{j,\mu}(y)\right),\\
&G_{i,\nu,j} = I_2\left(l_{i,\nu}(x)g_j^{(2)}(y)\right),\ H_{i,j,\mu} = I_2\left(l_i^{(2)}(x)h_{j,\mu}(y)\right),\\
&I_{i,\nu,j,\mu} = I_2\left(l_{i,\nu}(x)h_{j,\mu}(y)\right).
\end{aligned}
$$

Assuming that $w(x,y) = w_1(x)w_2(y)$ and that the partial moments

$$c_m = \int_a^{a'} w_1(x)x^m dx\,,\quad d_n = \int_b^{b'} w_2(y)y^n dy$$

exist, while $c_0 > 0, d_0 > 0$, we try to determine the points $x_i\big(i = \overline{1, m+1}\big)$ and $y_j\big(j = \overline{1, n+1}\big)$, so that we have

$$B_{i,j} = C_{i,j} = D_{i,j} = G_{i,\nu,j} = H_{i,j,\mu} = 0\,,$$

for any values of the parameters $\alpha_1, \ldots, \alpha_{m+1}, \beta_1, \ldots, \beta_{n+1}$.

It follows that $x_i\big(i = \overline{1, m+1}\big)$ and $y_j\big(j = \overline{1, n+1}\big)$ should be the real and distinct roots of the orthogonal polynomials of Christoffel-Szegö type, defined respectively by the formulas

$$
(3.5)\qquad U_{m+1}(x) = \frac{1}{u(x)}\begin{vmatrix}
P_{m+1}(x)P_{m+2}(x)\ldots P_{m+1+p}(x)\\
P_{m+1}(a_1)P_{m+2}(a_1)\ldots P_{m+1+p}(a_1)\\
P'_{m+1}(a_1)P'_{m+2}(a_1)\ldots P'_{m+1+p}(a_1)\\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
P_{m+1}^{(p_1-1)}(a_1)P_{m+2}^{(p_1-1)}(a_1)\ldots P_{m+1+p}^{(p_1-1)}(a_1)\\
P_{m+1}(a_2)P_{m+2}(a_2)\ldots P_{m+1+p}(a_2)\\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
P_{m+1}(a_r)P_{m+2}(a_r)\ldots P_{m+1+p}(a_r)\\
P'_{m+1}(a_r)P'_{m+2}(a_r)\ldots P'_{m+1+p}(a_r)\\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
P_{m+1}^{(p_r-1)}(a_r)P_{m+2}^{(p_r-1)}(a_r)\ldots P_{m+1+p}^{(p_r-1)}(a_r)
\end{vmatrix}
$$

$$(3.6)\qquad V_{n+1}(y)=\frac{1}{v(y)}\begin{vmatrix} Q_{n+1}(y)Q_{n+2}(y)\dots Q_{n+1+q}(y)\\ Q_{n+1}(b_1)Q_{n+2}(b_1)\dots Q_{n+1+q}(b_1)\\ Q'_{n+1}(b_1)Q'_{n+2}(b_1)\dots Q'_{n+1+q}(b_1)\\ \dots\dots\dots\dots\dots\dots\dots\dots\\ Q^{(q_1-1)}_{n+1}(b_1)Q^{(q_1-1)}_{n+2}(b_1)\dots Q^{(q_1-1)}_{n+1+q}(b_1)\\ Q_{n+1}(b_2)Q_{n+2}(b_2)\dots Q_{n+1+q}(b_2)\\ \dots\dots\dots\dots\dots\dots\dots\dots\\ Q_{n+1}(b_s)Q_{n+2}(b_s)\dots Q_{n+1+q}(b_s)\\ Q'_{n+1}(b_s)Q'_{n+2}(b_s)\dots Q'_{n+1+q}(b_s)\\ \dots\dots\dots\dots\dots\dots\dots\dots\\ Q^{(q_s-1)}_{n+1}(b_s)Q^{(q_s-2)}_{n+2}(b_s)\dots Q^{(q_s-1)}_{n+1+q}(b_s) \end{vmatrix}$$

where $p=p_1+\dots+p_{m+1}$, $q=q_1+\dots+q_{n+1}$ and $\{P_k\}$ is the system of orthogonal polynomials on (a,a') corresponding to the weight function $w_1(x)$, while $\{Q_k\}$ is the system of orthogonal polynomials on (b,b'), corresponding to the weight function $w_2(y)$.

If we chose for $x_1,\dots,x_{m+1}$ the roots of the polynomial $U_{m+1}(x)$ and for $y_1,\dots,y_{n+1}$ the corresponding roots of the polynomial $V_{n+1}(y)$, then we can conclude that the cubature formula (3.4) becomes

$$(3.7)\qquad \int_a^{a'}\int_b^{b'} w_1(x)w_2(y)f(x,y)dxdy=\sum_{i=1}^{m+1}\sum_{j=1}^{n+1}A_{i,j}f(x_i,y_j)+$$
$$+\sum_{i=1}^{n_1}\sum_{\nu=0}^{p_i-1}\sum_{j=1}^{n+1}E_{i,\nu,j}f^{(\nu,0)}(a_i,y_j)+\sum_{i=1}^{m+1}\sum_{j=1}^{n_2}\sum_{\mu=0}^{q_j-1}F_{i,j,\mu}f^{(0,\mu)}(x_i,b_j)+$$
$$+\sum_{i=1}^{n_1}\sum_{\nu=0}^{p_i-1}\sum_{j=1}^{n_2}\sum_{\mu=0}^{q_j-1}I_{i,\nu,j,\mu}f^{(\nu,\mu)}(a_i,b_j)+R(f).$$

We just proved the following theorem, which is the main result of this paper:

Theorem 3.2. *The cubature formula* (3.7) *has the degree of exactness* $(2m+p,2n+q)$.

Remark 3.1. Because this degree does not depend on the parameters α_i and β_j, we can consider the limit case $e(x)\equiv u(x)$, $g(y)\equiv v(y)$ in the remainder (2.2) and we will obtain the following result:

The remainder of the cubature formula (3.7) has the form

$$(3.8)\qquad R(f)=\iint_\Omega\left[(R_1f)(x,y)+(R_2f)(x,y)-(R_3f)(x,y)\right]w_1(x)w_2(y)dxdy\,,$$

where

$$(R_1f)(x,y)=u^2(x)A(x)\,[x,x_1,x_1,x_2,x_2,...,x_{m+1},x_{m+1},(a_1)^{p_1},...,(a_{n_1})^{p_{n_1}};f(t,y)]$$
$$(R_2f)(x,y)=v^2(y)B(y)\,[y,y_1,y_1,y_2,y_2,\dots,y_{n+1},y_{n+1},(b_1)^{q_1},\dots,(b_{n_2})^{q_{n_2}};f(x,z)]$$
$$(R_3f)(x,y)=u^2(x)v^2(y)A(x)B(y)\cdot$$
$$\cdot\begin{bmatrix} x,x_1,x_1,x_2,x_2,\dots,x_{m+1},x_{m+1},(a_1)^{p_1},\dots,(a_{n_1})^{p_{n_1}} \\ y,y_1,y_1,y_2,y_2,\dots,y_{n+1},y_{n+1},(b_1)^{q_1},\dots,(b_{n_2})^{q_{n_2}} \end{bmatrix};f(t,z)\,.$$

By applying the law of the mean to the integral (3.8) and the theorem of the mean for the divided differences, we obtain

$$\begin{aligned} R(f) &= \frac{C_1D_2}{(2m+2+p)!}\,f^{(2m+2+p,0)}(\xi,\eta_1)+\frac{C_2D_1}{(2n+2+q)!}\,f^{(0,2n+2+q)}(\xi_1,\eta)- \\ &- \frac{C_1C_2}{(2m+2+p)!(2n+2+q)!}\,f^{(2m+2n+4+p+q)}(\xi,\eta), \end{aligned}$$

where we have $\xi,\xi_1\in(a,a')$, $\eta,\eta_1\in(b,b')$ and

$$\begin{aligned} C_1 &= \int_a^{a'} w_1(x)A(x)P_{m+1}^2(x)dx, \quad C_2=\int_b^{b'} w_2(y)B(y)Q_{n+1}^2(y)dy, \\ D_1 &= \int_a^{a'} w_1(x)dx, \quad D_2=\int_b^{b'} w_2(y)dy. \end{aligned}$$

In the special case $w_1(x)\equiv w_2(y)\equiv 1$, $a=-1$, $a'=1$, $b=-1$, $b'=1$, $p_1=\dots=p_{n_1}=0$ and $q_1=\dots=q_{n_2}=0$, we obtain the Gauss bivariate cubature formula for the square $\Omega_2=[-1,1]\times[-1,1]$, namely

$$\iint_{\Omega_2} f(x,y)dxdy=\sum_{i=1}^{m+1}\sum_{j=1}^{n+1}A_{i,j}f(x_i,y_j)+R_{m,n}(f),$$

where we have

$$A_{i,j}=\frac{4}{(m+1)(n+1)P'_{m+1}(x_i)P'_{n+1}(y_j)P_m(x_i)P_n(y_j)}\,.$$

The remainder of this Gaussian cubature formula has the following expression

$$\begin{aligned} R_{m,n}(f) &= \frac{4^{m+2}}{2m+3}\cdot\frac{(m+1)!}{[(m+2)\dots(2m+2)]^3}\,f^{(2m+2,0)}(\xi,\eta_1)+ \\ &+\frac{4^{n+2}}{2n+3}\cdot\frac{(n+1)!}{[(n+2)\dots(2n+2)]^3}\,f^{(0,2n+2)}(\xi_1,\eta)- \\ &-\frac{4^{m+n+3}}{(2m+3)(2n+3)}\,\frac{(m+1)!(n+1)!}{[(m+2)...(2m+2)(n+2)...(2n+2)]^3}\,f^{(2m+2,2n+2)}(\xi,\eta), \end{aligned}$$

where by P_r is denoted the Legendre orthogonal polynomial of degree r, corresponding to the interval $[-1,1]$.

4. Illustrative examples

1) In the case of the weight functions

$$w_1(x) = (1-x)^{\alpha_1}(1+x)^{\beta_1}, \quad w_2(y) = (1-y)^{\alpha_2}(1+y)^{\beta_2}$$

where $\alpha_1, \beta_1, \alpha_2, \beta_2 > -1$, while the polynomials of the fixed nodes are $A(x) = 1 - x^2$, $B(y) = 1 - y^2$ and $m = n = 0$, we obtain the Gaussian nodes

$$x_1 = \frac{\beta_1 - \alpha_1}{\alpha_1 + \beta_1 + 4}, \quad y_1 = \frac{\beta_2 - \alpha_2}{\alpha_2 + \beta_2 + 4}.$$

For $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = -\frac{1}{2}$ we get the following cubature formula

$$\iint\limits_{\Omega_2} \frac{f(x,y)dxdy}{\sqrt{(1-x^2)(1-y^2)}} = \frac{\pi^2}{16}\Big\{f(-1,-1)+f(-1,1)+f(1,-1)+f(1,1)+$$

$$+2\left[f(-1,0)+f(0,-1)+f(1,0)+f(0,1)\right]+4f(0,0)\Big\}+R(f),$$

where in the case $f \in C^{4,4}(\Omega_2)$, the remainder is given by the formula

$$R(f) = -\frac{\pi^2}{192}\left[f^{(4,0)}(\xi,\eta_1) + f^{(0,4)}(\xi_1,\eta) + \frac{1}{192} f^{(4,4)}(\xi,\eta)\right].$$

2) If we consider the triple integral

$$I_3(f) = \iiint\limits_{\Omega_3} w(x,y,z)dx\,dy\,dz$$

extended on the 3-dimensional cube $\Omega_3 = [-1,1]^3$, and the weight function is $w(x,y,z) = w_1(x) = w_2(y) = w_3(z) = 1$, then for $m = n = p = 0$ we get the Gaussian cubature formula

$$\iiint\limits_{\Omega} f(x,y,z)dx\,dy\,dz = f\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right) + f\left(-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)+$$

$$+f\left(\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right) + f\left(-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)+$$

$$+f\left(-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right)+$$

$$+f\left(-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right) + R(f)$$

where the remainder has the expression

$$\begin{aligned} R(f) &= \frac{1}{135}\left[f^{(4,0,0)}(\xi,\eta',\zeta') + f^{(0,4,0)}(\xi',\eta,\zeta') + f^{(0,0,4)}(\xi',\eta',\zeta)\right] - \\ &- \frac{1}{18225}\left[f^{(4,4,0)}(\xi,\eta,\zeta') + f^{(4,0,4)}(\xi',\eta,\zeta') + f^{(0,0,4)}(\xi',\eta',\zeta)\right] + \\ &+ \frac{1}{9841500} f^{(4,4,4)}(\xi,\eta,\zeta). \end{aligned}$$

The degree of exactness of this cubature formula is $(3,3,3)$.

References

[1] Coman Gh., *Analiză numerică*, Libris, Cluj, 1995
[2] Davis P. J., Rabinowitz P., *Numerical Integration*, Blaisdell, Mass., 1967
[3] Ionescu D. V., *Cuadraturi numerice*, Editura Tehnică, Bucureşti, 1957
[4] Isaacson E., Keller H. B., *Analysis of Numerical Methods*, John Wiley, New York, 1966
[5] Stancu D. D., *Consideraţii asupra interpolării polinomiale a funcţiilor de mai multe variabile*, Bul. Univ. Babeş-Bolyai, Ser. Şt. Naturii **1**(1957), 43-82
[6] Stancu D. D., *Contribuţii la integrarea numerică a funcţiilor de mai multe variabile*, Acad. R. P. Rom., Fil. Cluj, Stud. Cerc. Matem., **8**(1957), 75-101
[7] Stancu D. D., *Asupra integrării numerice a funcţiilor de două variabile*, Acad. R. P. Rom., Fil. Iaşi, Stud. Cerc. Şt. Matem., **9**(1958), 5-21
[8] Stancu D. D., *Curs şi Culegere de probleme de Analiză Numerică*, Univ. Babeş-Bolyai, Cluj-Napoca, 1977
[9] Stancu D. D., Coman Gh., Blaga P., *Analiză numerică şi Teoria aproximării*, Presa Univ. Clujeană, 2002
[10] Taşcu, I., *Approximate cubature formulas with simple Gaussian nodes and with multiple fixed nodes*, BAM-volume, Techn. Univ. Budapest

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