CARPATHIAN J. MATH.
20 (2004), No. 2, 169 - 175

Generalized Inverses of Means

IULIA COSTIN

ABSTRACT. A mean N is called complementary to M with respect to P if it verifies the relation

$$P(M(a,b),N(a,b)) = P(a,b), \forall a,b > 0.$$

The complementary of M with respect to the geometric mean was called by C. Gini the inverse of M. We call the complementary of M with respect to a weighted geometric mean, generalized inverse of M. We study some generalized inverses, using the series expansion of means.

1. INTRODUCTION

Usually the means are given by the following

Definition 1.1. A **mean** is a function $M : \mathbb{R}_+^2 \to \mathbb{R}_+$, which has the property

$$\min(a,b) \le M(a,b) \le \max(a,b),\ \forall a,b > 0\ .$$

We use here weighted Gini means defined by

$$\mathcal{B}_{r,s;\lambda}(a,b) = \left[\frac{\lambda \cdot a^r + (1-\lambda)\cdot b^r}{\lambda \cdot a^s + (1-\lambda)\cdot b^s}\right]^{\frac{1}{r-s}}\ ,\ r \ne s$$

with $\lambda \in [0,1]$ fixed. Weighted Lehmer mean, $\mathcal{C}_{r;\lambda} = \mathcal{B}_{r,r-1;\lambda}$ and weighted power means $\mathcal{P}_{r,\lambda} = \mathcal{B}_{r,0;\lambda}$ are also used. We remark that

$$\mathcal{B}_{r,s;0} = \mathcal{C}_{r;0} = \mathcal{P}_{r,0} = \Pi_2 \text{ and } \mathcal{B}_{r,s;1} = \mathcal{C}_{r;1} = \mathcal{P}_{r,1} = \Pi_1\ ,$$

where we denoted by Π_1 and Π_2 the first respectively the second projections defined by

$$\Pi_1(a,b) = a,\ \Pi_2(a,b) = b,\ \forall a,b \ge 0.$$

Given three means M, N and P, their composition

$$P(M,N)(a,b) = P(M(a,b),N(a,b)),\ \forall a,b > 0,$$

defines also a mean $P(M,N)$.

Definition 1.2. A mean N is called **complementary to** M **with respect to** P (or $P-$**complementary** to M) if verifies

$$P(M,N) = P.$$

More comments on this notion and its importance can be found in [6]. We study the complementariness with respect to weighted geometric means $\mathcal{G}_\lambda = \mathcal{P}_{0,\lambda}$. We denote the $\mathcal{G}_\lambda-$ complementary of M by $M^{\mathcal{G}(\lambda)}$ and we call it *the generalized inverse of* M.

Received: 26.09.2004; In revised form: 10.01.2005
2000 *Mathematics Subject Classification.* 26E60.
Key words and phrases. *Generalized inverse of a mean, series expansion of a mean, complementary of a mean with respect to another, weighted Gini means.*

2. Series expansion of means

For the study of some problems related to means in [5] is used the power series expansion. In fact, for a mean M is considered the series of the normalized functions $M(1,1-x),\ x\in(0,1)$.

For example, in [3] is given the series expansion of the weighted Gini mean

$$\mathcal{B}_{p,p-r;t}(1,1-x)=1-(1-t)\cdot x+t\,(1-t)\,(2p-r-1)\cdot\frac{x^2}{2!}-t\,(1-t)$$
$$\cdot\,\{t[6p^2-6p\,(r+1)+(r+1)\,(2r+1)]-3p\,(p-r)-(r-1)\,(r+1)\}\cdot\frac{x^3}{3!}$$
$$-\,t\,(1-t)\cdot\{t^2[-24p^3+36p^2\,(r+1)-12p\,(r+1)\,(2r+1)+(r+1)\,(2r+1)$$
$$\cdot\,(3r+1)]++t[24p^3-12p^2\,(3r+1)+12p\,(r+1)\,(2r-1)-3\,(r+1)\,(2r+1)$$
$$\cdot\,(r-1)]--4p^3+6p^2\,(r-1)-2p\left(2r^2-3r-1\right)+(r-2)\,(r-1)\,(r+1)\}$$
$$\cdot\,\frac{x^4}{4!}--t\,(1-t)\cdot\left[t^3\left(120p^4-240p^3\,(r+1)+120p^2(r+1)(2r+1)\right.\right.$$
$$-20p(r+1)(2r+1)(3r+1)+(r+1)(2r+1)(3r+1)(4r+1))$$
$$+\,t^2\left(-180p^4+180p^3(2r+1)-90p^2(r+1)(4r-1)+30p(r+1)(2r+1)(3r-2)\right.$$
$$-6(r-1)(r+1)(2r+1)(3r+1))+t\left(70p^4-20p^3(7r-2)+10p^2(14r^2-6r-9)\right.$$
$$\left.-10p(r+1)(7r^2-12r+3)+(r-1)(2r+1)(7r-11)(r+1)\right)-5p^4+10p^3(r-2)$$
$$\left.-5p^2(2r^2-6r+3)+5p(r-2)(r^2-2r-1)-(r+1)(r-1)(r-2)(r-3)\right]\cdot\frac{x^5}{5!}+\cdots.$$

We need also the following result proved in [2].

Theorem 2.1. *If the mean M has the series expansion*

$$M(1,1-x)=1+\sum_{n=1}^{\infty}a_nx^n,$$

then the first terms of the series expansion of its generalized inverse $M^{\mathcal{G}(\lambda)}$ are

$$M^{\mathcal{G}(\lambda)}(1,1-x)=1-(1+\alpha\cdot a_1)\cdot x+\frac{\alpha}{2}\left[(\alpha+1)\cdot a_1^2+2\,(a_1-a_2)\right]\cdot x^2$$
$$-\,\frac{\alpha}{6}\left[(\alpha+1)\,(\alpha+2)\cdot a_1^3+3\,(\alpha+1)\cdot a_1\,(a_1-2a_2)+6\,(a_3-a_2)\right]\cdot x^3$$
$$+\,\frac{\alpha}{24}[(\alpha+1)\,(\alpha+2)\,(\alpha+3)\cdot a_1^4+4a_1^2\,(\alpha+1)\,(\alpha+2)\,(a_1-3a_2)$$
$$+\,12\,(\alpha+1)\left(a_2^2-2a_1\,(a_2-a_3)\right)+24\,(a_3-a_4)]\cdot x^4-\frac{\alpha}{5!}[(\alpha+1)\,(\alpha+2)$$
$$\cdot\,(\alpha+3)\,(\alpha+4)\cdot a_1^5+5a_1^3\,(\alpha+1)\,(\alpha+2)\,(\alpha+3)\,(a_1-4a_2)-60a_1^2$$
$$\cdot\,(\alpha+1)\,(\alpha+2)\,(a_2-a_3)+60a_1\,(\alpha+1)\left((\alpha+2)\,a_2^2+2\,(a_3-a_4)\right)+60a_2\,(\alpha+1)$$
$$\cdot\,(a_2-2a_3)-120\,(a_4-a_5)]\cdot x^5+\cdots$$

where

$$\alpha=\frac{\lambda}{1-\lambda}.$$

3. Generalized inverses of Gini means

As a consequence of the previous result, we get the following

Corollary 3.1. *The first terms of the series expansion of the generalized inverse of the Gini mean $\mathcal{B}_{p,p-r;\mu}$ are*

$$\begin{aligned}
&\mathcal{B}^{\mathcal{G}(\lambda)}_{p,p-r;\mu}(1,1-x) = 1-(\alpha\mu+\alpha-1)\cdot x-\alpha(1-\mu)\left[(\alpha+2p-r)\mu-(\alpha-1)\right]\cdot\frac{x^2}{2!}\\
&+\alpha(1-\mu)\{[3!\cdot p^2+6(\alpha-r)p+(\alpha-r)(\alpha-2r)]\mu^2-[3p^2-3(r-2\alpha)p\\
&+(2\alpha-r)(\alpha-r)]\mu+(\alpha-1)(\alpha+1)\}\cdot\frac{x^3}{3!}-\alpha(1-\mu)\{[4!\cdot p^3+36(\alpha-r)p^2\\
&+12(\alpha-r)(\alpha-2r)p+(\alpha-r)(\alpha-2r)(\alpha-3r)]\mu^3+[-24p^3+12(3r-4\alpha-1)p^2\\
&-12(2\alpha-2r+1)(\alpha-r)p-(\alpha-2r)(\alpha-r)(3\alpha+2-3r)]\mu^2+[4p^3+6(2\alpha-r\\
&+1)p^2+2\left(6\alpha(2\alpha-2r+1)-3r+2r^2-1\right)p+(\alpha-r)(3\alpha^2+4\alpha-3r\alpha-2r\\
&+r^2-1)]\mu-(\alpha-1)(\alpha+1)(\alpha+2)\}\cdot\frac{x^4}{4!}+\alpha(1-\mu)\{[5!\cdot p^4+240(\alpha-r)p^3\\
&+120(\alpha-r)(\alpha-2r)p^2+20(\alpha-r)(\alpha-2r)(\alpha-3r)p+(\alpha-r)(\alpha-2r)(\alpha-3r)\\
&\cdot(\alpha-4r)]\mu^4+[-180p^4+60(6r-7\alpha-2)p^3-90(\alpha-r)(3\alpha-4r+2)p^2\\
&-30(\alpha-r)\cdot(\alpha-2r)(2\alpha+2-3r)p-(\alpha-r)(\alpha-2r)(\alpha-3r)(4\alpha+5-6r)]\mu^3\\
&+[70p^4+20(10\alpha-7r+6)p^3+10\left(-30r\alpha+18\alpha^2+24\alpha+3+14r^2-18r\right)p^2\\
&+10(\alpha-r)\left(6\alpha^2+12\alpha-12r\alpha+7r^2-12r+3\right)p+(6\alpha^2-12r\alpha+15\alpha+5+7r^2\\
&-15r)(\alpha-2r)(\alpha-r)]\mu^2+[-5p^4+10(r-2-2\alpha)p^3+(30r\alpha-30\alpha^2-60\alpha\\
&-15-10r^2+30r\big)p^2-5(2\alpha+2-r)\left(2\alpha^2-2r\alpha+4\alpha-2r+r^2-1\right)p\\
&-(\alpha-r)\left(4\alpha^3-6r\alpha^2+15\alpha^2-15r\alpha+10\alpha+4r^2\alpha-5+5r^2-r^3-5r\right)]\mu\\
&+(\alpha-1)(\alpha+1)(\alpha+2)(\alpha+3)\}\cdot\frac{x^5}{5!}+\cdots.
\end{aligned}$$

Using it, we can prove the following

Theorem 3.2. *If*

$$\mathcal{B}^{\mathcal{G}(\lambda)}_{p,p-r;\mu}=\mathcal{B}_{q,q-s;\nu},\ \textit{for } r\neq 0, s\neq 0,$$

then we are in one of the following cases:

(i) $\mathcal{B}^{\mathcal{G}(0)}_{p,p-r;\mu}=\mathcal{B}_{q,q-s;0}$;
(ii) $\mathcal{B}^{\mathcal{G}(\lambda)}_{p,p-r;1}=\mathcal{B}_{q,q-s;0}$;
(iii) $\mathcal{B}^{\mathcal{G}\left(\frac{2}{3}\right)}_{p,-p}=\mathcal{B}_{q,q-s;1}$;
(iv) $\mathcal{B}^{\mathcal{G}}_{p,p-r;\mu}=\mathcal{B}_{-p,r-p;1-\mu}$;
(v) $\mathcal{B}^{\mathcal{G}\left(\frac{1}{3}\right)}_{p,p-r;0}=\mathcal{B}_{q,-q}$, *or*
(vi) $\alpha=\frac{\lambda}{1-\lambda}=\frac{\nu}{1-\mu},\ \mu=\frac{(2q-s)(1-\nu)}{r-2p}\ \ (r\neq 2p)$,
$\nu=\frac{q(q-s)\left(12p^2-12pr+5r^2\right)+(2p-r)(2q-s)\left(3p^2-3pr+r^2\right)+s^2\left(r^2-2pr+2p^2\right)}{2(rq-ps)(rq-rs+ps)},\nu\neq 1$

$rq \neq ps, rq + ps \neq rs, (36pr - 7r^2 - 36p^2)q^3(q - 2s) + 3(r - 2p)$
$\cdot(6p^2 - 6pr + r^2)q^2(2q - 3s) + (2r^2 - 7pr + 7p^2)s^2(p(r - p) + 2sq)$
$+(-72p^3r + 50p^2r^2 + 36p^4 - 14pr^3 + r^4)q(s - q)+$
$+(50pr - 9r^2 - 50p^2)s^2q^2 + (-2r^2s^3 + 3(r - 2p)(r^2 - 8pr + 8p^2)s^2)q$
$+p(r - p)s^4 + 3p(r - p)(r - 2p)s^3 = 0,$ *with* $s \neq 2q$.

The cases $(i) - (v)$ *are always true.*

Proof. Equating the coefficients of x in the series of the two members, we get:

$$\nu = \alpha (1 - \mu). \tag{3.1}$$

Then, the coefficients of x^2 give

$$\nu [(2q - s)(1 - \nu) + (2p - r)\mu] = 0 \tag{3.2}$$

If $\nu = 0$, then, from (3.1) we have one of the cases:
- $\alpha = 0$, therefore $\lambda = 0$, from where we obtain the case (i) , or
- $\mu = 1$, that leads to the case (ii).

If $\nu \neq 0$, then we have from (3.2):

$$(2q - s)(1 - \nu) = (r - 2p)\mu \tag{3.3}$$

1. If $r = 2p$, then one of the following situations must hold:
1.1. $\nu = 1$ that implies, from (3.1), $\mu = \frac{2\lambda - 1}{\lambda}$, with $\lambda \in \left(\frac{1}{2}, 1\right)$. Equating the coefficients of x^3, using the relation (3.1) and replacing $\nu = 1$ and $r = 2p$, it follows that:

$$p^2\mu(2\mu - 1) = 0.$$

We have again three possible situations:
1.1.1. $p = 0$, therefore $r = 0$, which contradicts with the hypothesis $r \neq 0$;
1.1.2. $\mu = 0$, so $\lambda = \frac{1}{2}$, therefore it is the case (iv), in which $r = 2p$ and $\mu = 0$;
1.1.3. $\mu = \frac{1}{2}$, so $\lambda = \frac{2}{3}$, thus the case (iii).
1.2. $2q = s$. Equating the coefficients of x^3 and taking into account the previous relations, the following condition is obtained:

$$p^2\mu(2\mu - 1) = (2\nu - 1)(\nu - 1)q^2. \tag{3.4}$$

1.2.1. In this relation, if $\nu = 1$, then one of the following situations must hold:
1.2.1.1. $\mu = 0$, then $\lambda = \frac{1}{2}$, therefore the case (iv), for $r = 2p$, $s = 2q$ and $\mu = 0$;
1.2.1.2. $\mu = \frac{1}{2}$, then $\lambda = \frac{2}{3}$, therefore the case (iii), for $s = 2q$;
1.2.1.3. $p = 0$, then $r = 0$, which contradicts with the hypothesis.
1.2.2. In the relation (3.4) if $\nu = \frac{1}{2}$, then again one of the following situations must hold:
1.2.2.1. $\mu = 0$, then $\lambda = \frac{1}{3}$, which leads to the case (v), for $r = 2p$;
1.2.2.2. $\mu = \frac{1}{2}$, then $\lambda = \frac{1}{2}$, that is the case (iv), for $r = 2p$ and $\mu = \frac{1}{2}$;
1.2.2.3. $p = 0$, then $r = 0$, which contradicts with the hypothesis $r \neq 0$.
1.2.3. If $\nu \neq 1$ and $\nu \neq \frac{1}{2}$ in the relation (3.4), then we can write:

$$q^2 = \frac{p^2\mu(2\mu - 1)}{(2\nu - 1)(\nu - 1)}. \tag{3.5}$$

This has real solutions if (a) $\mu \in \left[0, \frac{1}{2}\right]$ and $\nu \in \left(\frac{1}{2}, 1\right)$, or (b) $\mu \in \left[\frac{1}{2}, 1\right]$ and $\nu \in \left(0, \frac{1}{2}\right)$. In the case (a), we obtain $\lambda \in \left(\frac{1}{3-2\mu}, \frac{1}{2-\mu}\right) \subset \left(\frac{1}{3}, \frac{2}{3}\right)$. We continue with equating the coefficients of the reduced Taylor series of $\mathcal{B}^{\mathcal{G}(\lambda)}_{p,p-r;\mu}$ and $\mathcal{B}_{q,q-s;\nu}$. The coefficients of x^4 and the relations (3.1), $r = 2p$ and $s = 2q$ lead us to

$$(1+2\nu)\left(q^2 - 2p^2\mu^2 + p^2\mu - 3q^2\nu + 2q^2\nu^2\right) = 0,$$

which again is (3.4). Equating the coefficients of x^5 and taking into account the same relations and (3.5), we obtain the condition:

$$p^4\mu\,(2\mu-1)\,(\nu+\mu-1)\,(-10\mu+12\mu\nu+1-2\nu) = 0.$$

Hence, we have the following possible situations:
1.2.3.1. $p = 0$, then $q = 0$, but also $r = 0 = s$, which contradicts with the hypothesis;
1.2.3.2. $\mu = 0$, then (from (3.5)) $q = 0$, so $s = 0$, which contradicts with $s \neq 0$;
1.2.3.3. $\mu = \frac{1}{2}$, then (from (3.5)) $q = 0$, so $s = 0$, same as previous;
1.2.3.4. $\nu = 1-\mu$, then (from (3.5)) $q = \pm p$, and from (3.1), $\alpha = 1$, hence $\lambda = \frac{1}{2}$. This implies (iv), for $r = 2p$ and taking into account that $\mathcal{B}_{m,n;\gamma} = \mathcal{B}_{n,m;\gamma}$;
1.2.3.5. $-10\mu + 12\mu\nu + 1 - 2\nu = 0$, hence $\nu = \frac{10\mu-1}{12\mu-2}$, with $\mu \in \left[0, \frac{1}{10}\right] \cup \left[\frac{1}{2}, 1\right]$ (in order that $0 \le \nu \le 1$). But, for this value of ν, there is no real q from the relation (3.5) (none of the cases (a) and (b) are satisfied).
2. If $r \neq 2p$, from (3.3) we obtain:

$$\mu = \frac{(2q-s)\,(1-\nu)}{r-2p}. \tag{3.6}$$

Equating the coefficients of x^3 and taking into account (3.1) and (3.6), we obtain:

$$(1-\nu)\left[-2\,(rq-ps)\,(rq-rs+ps)\,\nu + \left(12p^2 - 12pr + 5r^2\right) q\,(q-s) + \right.$$

$$\left. + (r-2p)\left(3p^2 - 3pr + r^2\right)(s-2q) + \left(r^2 - 2pr + 2p^2\right) s^2\right] = 0 \tag{3.7}$$

2.1. If $\nu = 1$, from (3.6) it follows $\mu = 0$, hence, from (3.1), $\lambda = \frac{1}{2}$, which leads to the case (iv) for $\mu = 0$.
2.2. If $\nu \neq 1$, from (3.7) it follows that the following relation must hold:

$$2\,(rq-ps)\,(rq-rs+ps)\,\nu = -\left(12p^2 - 12pr + 5r^2\right) q\,(q-s) + \tag{3.8}$$

$$+ (r-2p)\left(3p^2 - 3pr + r^2\right)(s-2q) + \left(r^2 - 2pr + 2p^2\right) s^2 \tag{3.9}$$

2.2.1. If

$$rq = ps, \tag{3.10}$$

then the left side of the relation (3.9) is null, hence the right side must also be null. We have two possible situations:
2.2.1.1. $q = 0$. Then, from (3.10), $p = 0$ ($s \neq 0$), therefore the right side of (3.9) becomes

$$r^2 s\,(r+s) = 0.$$

Then, because $r \neq 0$ and $s \neq 0$, we have $r + s = 0$, which means $s = -r$, from where $\mu = 1-\nu$ and $\lambda = \frac{1}{2}$, obtaining the case (iv), when $p = 0$.
2.2.1.2. $q \neq 0$, then, from (3.10), $r = \frac{ps}{q}$. Replacing it in the right side of the relation (3.9) and equating with 0 we obtain:

$$p^2\,(p+q)\,(2q-s)^2\left(3q^2 - 3qs + s^2\right) = 0.$$

We can have one of the cases:
2.2.1.2.a. $p = 0$, thus $r = 0$, that cannot be fulfilled, because $r \neq 0$;
2.2.1.2.b. $p + q = 0$, thus $r = -s \Rightarrow$ (from (3.6)) $\mu = 1 - \nu \Rightarrow$ (from (3.1)) $\lambda = \frac{1}{2}$, therefore we obtained the case (iv);
2.2.1.2.c. $s = 2q$, or $r = \frac{ps}{q} = 2p$, that cannot be fulfilled, because we are in the case $r \neq 2p$; or
2.2.1.2.d. $3q^2 - 3qs + s^2 = 0$, that cannot be fulfilled unless $s = q = 0$, which contradicts with the hypothesis.
2.2.2. If in (3.9) we have $rq \neq ps$, but $rq - rs + ps = 0$, then

$$rq = s\,(r - p)\,.$$

We have $r \neq 0$, therefore

$$q = \frac{s\,(r - p)}{r}.$$

Replacing it in the right side of the relation (3.9) and equating with 0, we obtain

$$s\,(r - 2p)^2 \left(3p^2 - 3pr + r^2\right)(s - r) = 0.$$

But $s \neq 0$, $r \neq 2p$ and $3p^2 - 3pr + r^2 \neq 0$, hence the previous relation can be fulfilled only if $s = r$. In this case, we have $q = r - p$, therefore $\mu = 1 - \nu$ and $\lambda = \frac{1}{2}$, obtaining the case (iv), because $\mathcal{B}_{m,n;\gamma} = \mathcal{B}_{n,m;\gamma}$.
2.2.3. If in (3.9) we have $rq \neq ps$ and $rq - rs + ps \neq 0$, then we have:

$$\begin{aligned}(3.11)\quad \nu \;&= \left[q\,(q - s)\left(12p^2 - 12pr + 5r^2\right) + (2p - r)\left(3p^2 - 3pr + r^2\right)\cdot\right.\\ &\left.\cdot\,(2q - s) + s^2\left(r^2 - 2pr + 2p^2\right)\right]/\left[2\,(rq - ps)\,(rq - rs + ps)\right].\end{aligned}$$

Equating now the coefficients of x^4 and successively applying the relations (3.1), (3.6) and (3.11), we obtain:

$$\begin{gathered}(s - 2q)\,\{(36pr - 7r^2 - 36p^2)q^3(q - 2s) + 3(r - 2p)(6p^2 - 6pr + r^2)q^2\\ \cdot(2q - 3s) + (2r^2 - 7pr + 7p^2)s^2[p(r - p) + 2sq] + (-72p^3r + 50p^2r^2\\ +36p^4 - 14pr^3 + r^4)q(s - q) + (50pr - 9r^2 - 50p^2)s^2q^2 + [-2r^2s^3\\ +3(r - 2p)(r^2 - 8pr + 8p^2)s^2]qp(r - p)s^4 + 3p(r - p)(r - 2p)s^3\} = 0.\end{gathered}$$

2.2.3.1. If $s = 2q$, from (3.11) we obtain $\nu = \frac{1}{2}$, from (3.6) $\mu = 0$, hence $\alpha = \frac{1}{2}$, $\lambda = \frac{1}{3}$. Therefore, the case (v) is obtained.
2.2.3.2. If $s \neq 2q$, then the case (vi) of the conclusion has to be satisfied.
By direct computation it can be seen that the relations $(i) - (v)$ are verified. $\square$

Corollary 3.2. *The equality*

$$\mathcal{C}_{p,\mu}^{\mathcal{G}(\lambda)} = \mathcal{C}_{q,\nu}$$

is valid if and only if we are in one of the situations:

(i) $\mathcal{C}_{p,\mu}^{\mathcal{G}(0)} = \mathcal{C}_{q,0}$;
(ii) $\mathcal{C}_{p,1}^{\mathcal{G}(\lambda)} = \mathcal{C}_{q,0}$;
(iii) $\mathcal{C}_{\frac{1}{2}}^{\mathcal{G}\left(\frac{2}{3}\right)} = \mathcal{C}_{q,1}$;
(iv) $\mathcal{C}_{p,\mu}^{\mathcal{G}} = \mathcal{C}_{1-p,1-\mu}$, *or*
(v) $\mathcal{C}_{p,0}^{\mathcal{G}\left(\frac{1}{3}\right)} = \mathcal{C}_{\frac{1}{2}}$.

Proof. It follows from the previous theorem, by taking $r = s = 1$ and verifying that in the case (vi) there are no real solutions. $\square$

Corollary 3.3. *The equality*

$$\mathcal{B}^{\mathcal{G}(\lambda)}_{p,p-r;\mu} = \mathcal{P}_{q,\nu} \ \textit{(for } p \neq r, q \neq 0\textit{)}$$

is valid if and only if we are in one of the situations:

$$(i)\ \mathcal{B}^{\mathcal{G}(0)}_{p,p-r;\mu} = \mathcal{P}_{q,0};$$
$$(ii)\ \mathcal{B}^{\mathcal{G}(\lambda)}_{p,p-r;1} = \mathcal{P}_{q,0};$$
$$(iii)\ \mathcal{B}^{\mathcal{G}\left(\frac{2}{3}\right)}_{p,-p} = \mathcal{P}_{q,1}\ ,\ or$$
$$(iv)\ \mathcal{B}^{\mathcal{G}}_{p,0;\mu} = \mathcal{P}_{-p,1-\mu}.$$

Proof. It follows from the previous theorem, by taking $s = q$ and seeing that in the case (v) we would have $q = 0$, whereas in the case (vi) there are no real solutions. $\square$

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Technical University of Cluj-Napoca
Department of Computer Science
Bariţiu 28, 400027, Cluj-Napoca, Romania
E-mail address: Iulia.Costin@cs.utcluj.ro