CARPATHIAN J. MATH.
20 (2004), No. 2, 187 - 192

Localization of solutions for a problem arising in the theory of adiabatic tubular chemical reactors

ANDREI HORVAT-MARC and CRISTINA ŢICALĂ

ABSTRACT. We consider the boundary value problem

$$\begin{cases} \mu u'' - u' + f(u) = 0, \text{ on } [0,1] \\ \mu u'(0) - u(0) = 0 \\ u'(1) = 0 \end{cases}$$

where μ is a positive real number and $f : \mathbb{R} \to \mathbb{R}$ is continuous.

For this problem, via Krasnoselskii expansion-compression theorem, we establish an existence result for positive solutions and we use the localization provided by the Theorem 2.3 to give an aproximation of the solution for some particular cases.

1. INTRODUCTION

In this paper we establish existence conditions of positive solutions of boundary value problem

$$\begin{cases} \mu u'' - u' + f(u) = 0, \quad \text{on } [0,1] \\ \mu u'(0) - u(0) = 0 \\ u'(1) = 0 \end{cases} \tag{1.1}$$

This work was inspired by [2], where the problem (1.1) is studied using some results related to fixed point index. Our approach use the Krasnoselskii expansion-compression theorem.

The Krasnoselskii expansion-compression theorem has been improved in various way [2, 3, 4, 10] and applied to establish sufficient conditions for existence of positive solutions of different problems [1, 5, 7, 9]. In recent paper [6] this technique is applied to discuss the nonlinear integral equations in Banach spaces.

In what follows, we recall the Krasnoselskii expansion-compression theorem.

Theorem 1.1. [Krasnoselskii expansion-compression fixed point theorem] *Let X be a Banach space, and let $K \subset X$ be a con in X. Assume that Ω_1, Ω_2 are two open subsets of X such that $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Consider the operator $T : K \cap \left(\overline{\Omega}_2 \backslash \Omega_1\right) \to K$ be completely continuous and either*

$$\|T(x)\| \le \|x\|, \ x \in K \cap \Omega_1 \text{ and } \|T(x)\| \ge \|x\|, \ x \in K \cap \Omega_2$$

or

$$\|T(x)\| \ge \|x\|, \ x \in K \cap \Omega_1 \text{ and } \|T(x)\| \le \|x\|, \ x \in K \cap \Omega_2$$

is true. Then T has a fixed point in $K \cap \left(\overline{\Omega}_2 \backslash \Omega_1\right)$.

Received: 30.09.2004; In revised form: 05.12.2004
2000 *Mathematics Subject Classification.* 47H09, 47H99.
Key words and phrases. *Krasnoselskii expansion-compression theorem, positive solutios, boundary value problem.*

In this section we remind an extension of Krasnoselskii expansion-compression theorem presented in [5]. This result is the main tool in our approach.

We denote by $C([0,h], \mathbb{R}_+)$, where $\mathbb{R}_+ = [0,\infty)$, the Banach space of continuous functions $u : [0,h] \to \mathbb{R}_+$, endowed with the Chebyshev norm

$$\|u\| = \max_{t\in[0,h]} \{|u(t)|\}.$$

Let us consider the Fredholm integral equation

$$u(t) = g(t) + \int_0^h k(t,s)\, f(s,u(s))ds, \quad t \in [0,h] \tag{1.2}$$

with $g \in C(0,h)$ and $f : [0,h] \times [0,\infty) \to \mathbb{R}$ continuous.

We introduce the following conditions:

(H_1) $0 \le k_t(s) = k(t,s) \in L^1[0,h]$ for any $t \in [0,h]$;
(H_2) the map $t \longmapsto k_t$ is continuous from $[0,h]$ to $L^1[0,h]$;
(H_3) there are $\mu \in (0,1)$, $\kappa \in L^1[0,h]$, and an interval $[a,b] \subset [0,h]$ such that $k(t,s) \ge \mu\kappa(s) \ge 0$ for any $t \in [0,h]$ and a.e. $[0,h]$;
(H_4) $k(t,s) \le \kappa(s)$ for any $t \in [0,h]$ and a.e. $s \in [0,h]$;
(H_5) $g \in C[0,h]$ with $g(t) \ge 0$ for any $t \in [0,h]$ and

$$\min_{a\le t\le b} g(t) \ge \mu \|g\| = \mu \sup_{0\le t\le h} |g(t)|;$$

(H_6) $f : [0,h] \times \mathbb{R} \to \mathbb{R}$ is continuous;
(H_7) there are the nondecreasing mappings $\varphi, \psi \in C(\mathbb{R}_+, \mathbb{R}_+)$ and the positive numbers σ, τ such that

$$0 \le f(t,z) \le \psi(z) \text{ for any } t \in [0,h] \text{ and a.e. } z \in [0,\sigma],$$

$$0 < \varphi(z) \le f(t,z) \text{ for any } t \in [0,h] \text{ and a.e. } z \in [0,\tau];$$

(H_8) there exists $\alpha > 0$ such that $\alpha < \sigma$ and $\frac{\alpha}{\|g\|+K_1\psi(\alpha)} > 1$, where

$$K_1 = \sup_{0\le t\le h} \int_0^h k(t,s)ds > 0;$$

(H_9) there exist $\beta > 0, \alpha \ne \beta$, $\beta < \tau$ and $t_0 \in [0,h]$ such that

$$\frac{\beta}{g(t_0) + \varphi(\mu\beta)\int_a^b k(t_0,s)ds} < 1.$$

Theorem 1.2. [5] *Assume that hypothesis* $(H_1) - (H_9)$ *are satisfied. Then, equation* (1.2) *has at least one nonnegative solution* $u \in C[0,h]$ *and for this solution we have that*

$$\min\{\alpha,\beta\} \le \|u\| \le \max\{\alpha,\beta\}.$$

In what follows we present another existence result.

Lemma 1.1. *If* $0 < \alpha < \beta$, $g \in C[0,h]$ *and hypotheses* (H_1), (H_2), (H_6), (H_7) *and* (H_8) *are satisfied, then there exists a positive solution* $u_0 \in C\,[0,h]$ *for the nonlinear integral equation* (1.2) *such that*

$$0 < \|u_0\| \leq \alpha. \tag{1.3}$$

Proof. Consider the function $f^* : \mathbb{R} \to \mathbb{R}$

$$f^*(t) = \begin{cases} f(t)\,, & \text{if } t \geq 0 \\ f(0)\,, & \text{if } t < 0 \end{cases}$$

and the complete continuous operator $T^* : C\,([0,h]\,,\mathbb{R}_+) \to C\,([0,h]\,,\mathbb{R}_+)$,

$$(T^*u)(t) = g(t) + \int_0^h k(t,s)\, f^*(u(s))\, ds, \;\; t \in [0,h]\,.$$

Let $u \in C\,([0,h]\,,\mathbb{R}_+)$ be a solution of $u(t) = \lambda\,(T^*u)(t)$, $t \in [0,h]$ for some $\lambda \in (0,1)$. We have

$$\|u\| = \sup_{t\in[0,h]} u(t) \leq \|g\| + \psi(\|u\|) \cdot \sup_{t\in(0,h)} \int_0^h k(t,s)\, ds.$$

By (H_8) we obtain that $\|u\| = \alpha$ and this implies the existence of a positive solution $u_0 \in C\,[0,h]$ such that $0 < \|u_0\| \leq \alpha$. $\square$

2. Main result

We consider the boundary value problem

$$\begin{cases} \mu u'' - u' + f(u) = 0, \;\text{ on }\; [0,1] \\ \mu u'(0) - u(0) = 0 \\ u'(1) = 0 \end{cases} \tag{2.4}$$

where μ is positive real number and $f : [0,\infty) \to \mathbb{R}$ is continuous. This problem arises in the theory of adiabatic tubular chemical reactors, where u represents the temperature.

In some particular cases [2], function $f : \mathbb{R}^+ \to \mathbb{R}$ represents the *Arrhenius reaction rate*

$$f(t) = p\,(q-t)\, e^{-\dfrac{c}{1+t}}, \;\text{ with } p, q, c \text{ pozitive.} \tag{2.5}$$

It is easy to chek that problem (2.4) is equivalent to the nonlinear integral equation

$$u(t) = \int_0^1 G(t,s)\, f(u(s))\, ds, \;\; t \in [0,1] \tag{2.6}$$

where the Green function $G : [0,1] \times [0,1] \to [0,1]$ is given by

$$G(t,s) = \begin{cases} 1 & \text{if} \quad 0 \leq s \leq t \leq 1 \\ e^{\frac{t-s}{\mu}} & \text{if} \quad 0 \leq t \leq s \leq 1 \end{cases}. \tag{2.7}$$

For the Green function given by (2.7) we have

Lemma 2.2. (LG_1) $\sup\limits_{t\in[0,1]} \int_0^1 G(t,s)\, ds = 1$;

(LG_2) *for* $t_0 \in [a,b]$ *the following inequalities hold*

$$\mu\left(1-e^{\frac{a-b}{\mu}}\right) \le \int_a^b G(t_0,s)\,ds = t_0+\mu-a-\mu e^{\frac{t_0-b}{\mu}} \le b-a. \tag{2.8}$$

In order to apply Theorem 1.2 we need the next intermediate results

Lemma 2.3. *Consider the function* $\gamma : [0,1] \to [0,1]$, $\gamma(t) = e^{\frac{t}{\mu}}$. *Then*
(L_1) $G(t,s) \le \gamma(s)$ *for* $t \in [0,1]$ *and a.e.* $s \in [0,1]$;
(L_2) *for any* $0 \le a < b \le 1$ *we have*

$$G(t,s) \ge e^{\frac{a-2}{\mu}}\gamma(s),\ t \in [a,b] \ \text{ and } s \in [0,1]. \tag{2.9}$$

Proof. (L_1) It's obviously that $G(t,s) \le 1 \le \gamma(s) = e^{\frac{t}{\mu}}$ for $t,s \in [0,1]$.
(L_2) Let $[a,b] \subset [0,1]$. If $s \in [0,1]$, $t \in [a,b]$ with $s \le t$ then (2.9) is equivalent with $1 \ge e^{\frac{a+s-2}{\mu}} \ge e^{\frac{a-2}{\mu}}$ and this is true because $a+s-2 \le 0$. If $s \in [0,1]$, $t \in [a,b]$ with $s \ge t$ then (2.9) is equivalent with $e^{\frac{t-s}{\mu}} \ge e^{\frac{a+s-2}{\mu}} \ge e^{\frac{a-2}{\mu}}$ and this is true because $t-a \ge 0 \ge 2s-2$. □

The main result of this paper is given in the following theorem

Theorem 2.3. *Suppose that the function* $f : \mathbb{R} \to \mathbb{R}$ *is continuous and there exist the positive numbers* α, β $(\alpha \ne \beta)$, σ, τ *with* $\alpha < \sigma$, $\beta < \tau$ *and the nondecreasing continuous functions* $\varphi, \psi \in C(\mathbb{R}_+, \mathbb{R})$ *such that*

$$0 \le f(z) \le \psi(z) < \alpha \ \text{ a.e. in } z \in [0,\sigma], \tag{2.10}$$

$$0 < \varphi(z) \le f(z) \ \text{ a.e. in } z \in [0,\tau]; \tag{2.11}$$

$$\beta < \mu\left(1-e^{\frac{a-b}{\mu}}\right)\varphi\left(\beta e^{\frac{a-2}{\mu}}\right) \quad \text{for some } a,b \in [0,1], a < b. \tag{2.12}$$

Then, the problem (2.4) *have at least one positive solution* $u \in C[0,1]$. *Furthermore*

(A) $\alpha \le \|u\| \le \beta$ *and* $u(t) \ge \alpha e^{\frac{a-2}{\mu}}$ *for* $t \in [a,b]$ *if* $\alpha < \beta$,

(B) $\beta \le \|u\| \le \alpha$ *and* $u(t) \ge \beta e^{\frac{a-2}{\mu}}$ *for* $t \in [a,b]$ *if* $\beta < \alpha$,

holds.

Proof. The proof is a direct application of Theorem 1.2 with $k = G$, $\kappa = \gamma$ and $g \equiv 0$. Hence (H_3) and (H_4) are equivalent with (L_2), respectively (L_1). For our considerations, (H_8) implies $\psi(z) < \alpha$, $z \in [0,\sigma]$ inequality which are impose in (2.10). By (LG_2) we obtain that (2.12) implies (H_9). So, we can apply Theorem 1.2 to obtain the existence of positive solution for (2.4). □

Remark that (2.12) can be replaced by

$$\beta < \left(t_0+\mu-a-\mu e^{\frac{t_0-b}{\mu}}\right)\varphi\left(\beta e^{\frac{a-2}{\mu}}\right) \ \text{ for some } 0 \le a < t_0 < b \le 1. \tag{2.13}$$

3. Aplication and numerical results

In this section we obtain some numerical results using Theorem 2.3.
Consider the problem

$$(3.14)\qquad \begin{cases} u'' - u' + \left(-\frac{u}{2}+1\right)e^{-\frac{1}{1+u}} = 0, & \text{on } [0,1] \\ u'(0) - u(0) = 0 \\ u'(1) = 0 \end{cases}$$

Problem (3.14) is a particular case of (2.4) with

$$f(x) = \left(-\tfrac{x}{2}+1\right)e^{-\frac{1}{1+x}},\ \ x \in [0,\infty).$$

For $x \in [0,1]$ we have $0.3 < f(x) < 0.4$ and therefore let $\psi(x) \equiv 0.4$ and $\varphi(x) \equiv 0.3$. We choose $\alpha = 0.4$ and $\beta = 0.1$. These values satisfi (2.10) and (2.11). Hypothesis (2.12) is satisfied if $b - a > 0.40547$. So, we can conclude that for (3.14) there exists at least one positive solution $u \in C([0,1],[0,1])$ and for this solution we have

$$0.1 \le u(t) \le 0.4,\ \ t \in [0,1]$$

$$u(t) \ge 0.1 \cdot e^{a-2}, t \in [a,b] \ \text{ with } a \in \left[0,\tfrac{1}{2}\right], b \in \left[a+\tfrac{1}{2},1\right]$$

Let $Lu(t) = u''(t) - u'(t) + f(u(t))$, $t \in [0,1]$. For the first iteration u_0 we used several values.

Considering $u_0 = 0.3$, it results

$$\begin{aligned} u_1(t) &= \int_0^1 G(t,s)\, f(u_0(s))\, ds \\ &= \int_0^1 G(t,s)\, f(0.3)\, ds \\ &= 0.393\,86t - 0.393\,86e^{t-1} + 0.393\,86, \end{aligned}$$

where u_1 satisfies the boundary conditions and

$$-0,00195 < Lu_1(t) < 0.0000152, t \in [0,1].$$

Considering $u_0 = 0.25$, it results

$$\begin{aligned} u_1(t) &= \int_0^1 G(t,s)\, f(u_0(s))\, ds \\ &= \int_0^1 G(t,s)\, f(0.25)\, ds \\ &= 0.393\,16t - 0.393\,16e^{t-1} + 0.393\,16, \end{aligned}$$

where u_1 satisfies the boundary conditions and

$$-0.0013 < Lu_1(t) < 0.0007152, t \in [0,1].$$

Considering $u_0 = 0.2$, it results

$$\begin{aligned} u_1(t) &= \int_0^1 G(t,s) f(u_0(s))\, ds \\ &= \int_0^1 G(t,s) f(0.2)\, ds \\ &= 0.391\,14t - 0.391\,14e^{t-1} + 0.391\,14, \end{aligned}$$

where u_1 satisfies the boundary conditions and

$$0.\,00087 \times 10^{-4} < Lu_1(t) < 0.00272, t \in [0,1].$$

References

[1] R. Agarwal, M. Meehan, D. O'Regan & R. Precup, *Location of nonnegative solutions for differential equation on finite and semi-infinite intervals*, Dynamic System Appl. **12** (3-4) (2003), 323-341

[2] K. Deimling, *Nonlinear Functional Analysis*, Springer, Berlin, 1985.

[3] K. Deimling, *Positive Fixed Points of Weakly Inward Maps*, Nonlinear Analysis, Theory & Applications, **12** (1988), 223–226

[4] K. Deimling, *Fixed Point of Weakly Inward Maps in Conical Shells*, Nonlinear Analysis, Theory & Application, **12** (1988), 227–230

[5] A. Horvat-Marc, *Nonnegative Solutions for Boundary Value Problem* Bul. Ştiinţ. Univ. Baia Mare, **18**(2002), 223 - 228

[6] A. Horvat-Marc & Radu Precup, *Nonnegative solutions of nonlinear integral equations in ordered Banach spaces*, Fixed Point Theory, Vol. **5** (2004), No. 1, 65-70

[7] G. L. Karakostas & P. Ch. Tsamatos, *Multiple positive solutions of some Fredholm integral equations arisen from nonlocal boundary-value problems*, Electronic Journal of Differential Equations, **2002** (2002), No. 30, 1 - 17

[8] M. A. Krasnoselskii, *Positive solutions of operator equations*, Noordhoff, Groningen, 1964

[9] M. Meehan & D. O'Regan, *Positive L^p solutions of Hammerstein integral equation*, Arch. Math., **76** (2001), 366-376

[10] R. Precup, *On Some Fixed Point of Deimling*, Nonlinear Analysis, Theory Methods & Applications, **23** (1994), 1315–1320

North University of Baia Mare
Department of Mathematics and Computer Science
Victoriei 76, 430122, Baia Mare,
Romania
E-mail address: hmandrei@rdslink.ro
E-mail address: cristina_ticala_p@yahoo.com