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About the Dirichlet and Neumann boundary value problems expressed by means of monogenous quaternions

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ABSTRACT. Using the solution of Dirichlet's plane problem, the solution given as a monogenous surface, we obtain the solution of Neumann's plane problem through the rotation of the quaternion associated to the solution of Dirichlet's problem.

1. INTRODUCTION

In [1] and [2] we introduced the monogenous quaternion given by the following relation:

$$K = x + iy + ju(x, y) + kv(x, y) \tag{1.1}$$

where $(1, i, j, k)$ are from the A4 algebra, associative, but non-commutative ($ij = -ji = k$; $jk = -kj = i$; $ki = -ik = j$; $i^2 = j^2 = k^2 = -1$).

A real surface of form (S) was associated

$$\bar{r} = \bar{i}y + \bar{j}u(x, y) + \bar{k}v(x, y) \tag{1.2}$$

where $(\bar{i}, \bar{j}, \bar{k})$ are the usual Euclidean versors.

We demonstrated that the monogenity of the quaternion given be the relation (1.1) induces the associated surface given in relation (1.2) the following properties:

1. the monogenous surface (1.2) is a minimal surface (has a null medium curvature)
2. the monogenous surface (1.2) has a negative Gauss curvature.

In articles [3] and [4] we found some more properties for surfaces of the form (1.2), namely:

Theorem 1.1. *The minimal surface having a negative total curvature is the solution of a Dirichlet plane problem of the form: Find $u(x, y)$ if*

$$\begin{cases} \Delta u = 0 \text{ in } D \subset \mathbb{R}^2 \\ u|_C = u_0(\zeta),\ \zeta \in C,\ C = Fr\, D, \end{cases} \tag{1.3}$$

($Fr\, D$ is boundary of D, supposed to be smooth).

The determination of the minimal surface, solution of the Dirichlet problem, will be done within the following equivalent problem:

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2. The Dirichlet equivalent problem

Let us determine a minimal surface with a negative total curvature knowing a curvature of coordinates belonging to it. The Neumann boundary value problem is known as:

Let us find the function:

$$u(x,y) : D \subset \mathbb{R}^2 \to \mathbb{R} \tag{2.4}$$

such that

$$\begin{cases} \Delta u = 0 \text{ in } \quad D, \text{simple conex} \\ \left.\dfrac{\partial u}{\partial n}\right|_C = u_1(\zeta), \zeta \in C = Fr\, D, \end{cases} \tag{2.5}$$

(Fr, D = boundary of D, supposed to be smooth.)

Because $\dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial s} = u_1(\zeta)$, noting $v_0(\zeta) = \int_0^s u_1(\zeta)\, ds$, $\zeta \in C$, we will be able to express Neumann's problem as follows:

Let us determine a function $v(x,y) : D \subset \mathbb{R}^2 \to \mathbb{R}$, knowing that

$$\begin{cases} \Delta v = 0 \text{ in } D, \text{ simple conex} \\ v|_C = v_0(\zeta) \ = \ \int_0^s u_1(\zeta)\, d\zeta \end{cases} \tag{2.6}$$

where $v(x,y)$ is the harmonic conjugate of function $u(x,y)$.

Remark 2.1. The solution of the problem is unique if:

$$\oint_C u_1(\zeta)\, d\zeta = 0.$$

Thus expressed, the Neumann problem is a Dirichlet problem for the function $v(x,y)$– the harmonic conjugate of $u(x,y)$. In the particular case when the domain D is the disk $|\, z\,| < a$, we note:

$$\begin{matrix} u^* = Re\{zw^/(z)\} = xu_x + yu_y \\ v^* = Im\{zw^/(z)\} = yu_x - xu_y \end{matrix} \tag{2.7}$$

If the solution $w(z) = u + iv$ is known, obtained with an approximation to a constant trough knowing the solution $u(x,y)$ of Dirichlet problem, then $w^* = u^* + iv^*$is the solution obtained from the solution of the Neumann problem for the same disk.

We express in the language of quaternions the Dirichlet and Neumann limit problem for a disk, as well as their solutions.

We associate quaternion K of form (1.1) to the solution of problem (2.4), and surface (S) of form (1.2) to the solution of the Dirichlet plane problem for the disk. In the same way, we associate a surface (S^*) and a quaternion K^* of the following form (2.8), to the solution of Neumann's plane problem for the same disc:

$$\begin{matrix} (S^*) : \bar{r} = \bar{i}y + \bar{j}u^*(x,y) + \bar{k}v^*(x,y)\, ; \quad (\bar{i}, \bar{j}, \bar{k}) \text{ euclidian versors} \\ K^* = x + iy + ju^*(x,y) + kv^*(x,y)\, ; \quad (1, i, j, k) \in A_4 \end{matrix} \tag{2.8}$$

Surface (S^*) is minimal, has a negative total curvature and represents the solution to the Neumann problem. We will demonstrate the following theorem:

Theorem 2.2. *Any surface S^* of the form (2.8) associated to the quaternion K^*, where $u^* = xu_x + yu_y$; $v^* = yu_y - xu_y$ is a minimal surface with a negative total curvature, if surface (S) of the form (1.2) is monogenous.*

Proof. Indeed, calculating $\dfrac{\partial K^*}{\partial z}$ and $\dfrac{\partial K^*}{\partial \bar{z}}$ we obtain

$$(2.9)\quad \begin{cases} \dfrac{\partial K^*}{\partial z} = \dfrac{1}{2}\left(\dfrac{\partial K^*}{\partial x} - i\dfrac{\partial K^*}{\partial y}\right) = \\ \qquad = (1, 0, u_x + yu''_{xy} + u''_{x^2}, -xu''_{xy} - u_y + yu''_{y^2}) = (1, 0, P, Q) \\ \dfrac{\partial K^*}{\partial \bar{z}} = (0, 0, xu''_{xy} + u_y + yu''_{y^2}, u_x + yu''_{xy} - xu''_{xy}) \end{cases}$$

We notice that

$$(2.10)\quad \begin{array}{l} \dfrac{\partial K^*}{\partial z} = (1, 0, P, Q) \\ \dfrac{\partial K^*}{\partial \bar{z}} = 0 \end{array}$$

We notice that the monogenity conditions Cauchy-Riemann $\dfrac{\partial P}{\partial x} = \dfrac{\partial Q}{\partial y}$, $\dfrac{\partial P}{\partial y} = -\dfrac{\partial Q}{\partial x}$ take place if and only if $\Delta u = 0$.

Under these conditions, surface (S^*) associated to the monogenous quaternion K^* is a minimal surface of a negative Gauss curvature. □

3. Quaternions and space rotations

We remark first that if $A = a + bi + cj + dk$ any given quaternion, then $K^* = AK$ will lead to the linear system:

$$(3.11)\quad \begin{cases} x = xa - yb - uc - vd \\ y = ya + xb - vc + ud \\ u^* = ua + vb + xc - yd \\ v^* = va - ub + yc + xd \end{cases}$$

The determinant of the system is $\Delta = (x^2 + y^2 + u^2 + v^2)^2 \neq 0$ or $[|\, z\,|^2 + |\, w(z)\,|^2]^2 \neq 0$ where $w = u + iv$ and we obtain a unique quaternion $A = a + bi + cj + dk$ with a , b, c, d, from the system (3.11).

It is known [5] that one can associate a space rotation to any quaternion. The result given by $K^* = AK$ can be interpreted as the multiplication result of two space rotations $A \cdot K$.

If K^* – the quaternion associated to a monogenous surface, solution of the Neumann problem, and K – the quaternion associated to a surface – solution of the Dirichlet problem, we will have

Theorem 3.3. *The solution K^* (the quaternion associated to the solution of the Neumann problem) for a plane problem (2.6) in the disk $|\, z\,| < a$ is obtained from* *the solution K – the quaternion associated to the Dirichlet problem for the same disk, by a space rotation given by the quaternion $A = a + bi + cj + dk$.*

Proof. Indeed, using system (3.11), we replace $u^* = xu_x + yu_y$; $v^* = yu_x - xu_y$ we obtain the unique values of a, b, c, d, given by the linear system (3.12)

$$\left\{ \begin{array}{l} x = xa - yb - uc - vd \\ y = ya + xb - vc + ud \\ xu_x + yu_y = ua + vb + xc - yd \\ yu_x - xu_y = va - ub + yc + xd \end{array} \right. \tag{3.12}$$

$a = \dfrac{\Delta a}{\Delta}$, $b = \dfrac{\Delta b}{\Delta}$, $c = \dfrac{\Delta c}{\Delta}$, $d = \dfrac{\Delta d}{\Delta}$, with $\Delta = (x^2 + y^2 + u^2 + v^2)^2 \neq 0$. □

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